# HALF-INTEGRAL WEIGHT KLOOSTERMAN SUMS AND INTEGER PARTITIONS

BY

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DISSERTATION

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# Abstract


Kloosterman sums are special exponential sums which appear in many problems in number theory. Kloosterman first introduced these sums in [1] to investigate whether the quadratic form $a_1n_1^2 + a_2n_2^2 + a_3n_3^2 + a_4n_4^2$ with fixed $a_i \in \mathbb{N}$ represents all sufficiently large natural numbers. Another application is to estimate the shifted sum of divisor functions. Let $\tau(n)$ be the number of divisors of the positive integer $n$ and

$$D(N, f) := \sum_{n=1}^{N} \tau(n)\tau(n+h), \quad \text{for some fixed integer } h \geq 1.$$

Heath-Brown [2] applied the Weil bound (1.1) of Kloosterman sums to prove that

$$D(N, f) = \text{explicit main terms} + O(N^{\frac{5}{6}+\varepsilon}), \quad \text{uniformly for } 1 \leq h \leq N^{\frac{5}{6}}.$$

Using Kuznetsov's trace formula, Deshouillers and Iwaniec [3] obtained a much better error bound $O(N^{\frac{2}{3}+\varepsilon})$ for all $h \geq 1$.

The integer partition function $p(n)$, which is the number of ways to write $n$ as a sum of positive integers, has been researched for remarkable properties by Euler, Hardy and Ramanujan [4]. Rademacher's exact formula [5] states that $p(n)$ can be written as a sum of exponential sums. The generating function of $p(n)$ is $q^{\frac{1}{24}}/\eta(z)$, where $\eta(z)$ is Dedekind's eta function with $q = e^{2\pi i z}$ and $\operatorname{Im} z > 0$. Since $\eta(z)$ is a weight $\frac{1}{2}$ modular form, using the definition of multiplier systems, we are able to rewrite the exponential sums in Rademacher's exact formula as generalized Kloosterman sums. The bounds on Kloosterman sums give the growth rate of errors for such approximations.

There are very famous congruence properties of the partition function $p(n)$ by Ramanujan:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}.$$

In 1944, Dyson [6] defined the rank of a partition of $n$. If we let $N(a, b; n)$ denote the number of partitions of $n$ with rank congruent to $a \pmod b$, then Dyson conjectured that $5N(j, 5; 5n+4) = p(5n+4)$ and $7N(j, 7; 7n+5) = p(7n+5)$ for all $j$. By the work of Bringmann and Ono [7], [8], the generating functions for the ranks of partitions have similar properties as $q^{\frac{1}{24}}/\eta(z)$. The work of Bringmann and Ono in the theory of harmonic Maass forms discovers beautiful properties about the rank of partitions. For example, in [7] they proved the exact formula for the modulo 2 case, which perfected the asymptotics by Ramanujan, Dragonette [9] and Andrews [10].

If we have better estimates for the sums of half-integral weight Kloosterman sums, we are able to obtain better tail bounds for the Rademacher-type exact formulas, which control the efficiency of their convergence. The recent work by Ahlgren and Andersen [11], Ahlgren and Dunn [12], and Andersen and Wu [13] provide

improved error bounds based on their improvement on the estimates for Kloosterman sums.

The author [14], [15] generalized their work to the Kloosterman sums with a wider class of multiplier systems, which are half-integral weight and include the commonly used theta- and eta-multipliers twisted by quadratic characters. The resulting estimates give a uniform version of the general result by Goldfeld and Sarnak [16] for sums of such Kloosterman sums with a power-saving bound in the parameters $m$ and $n$. Following the method in [7], the author provided a detailed proof of the exact formula for the rank modulo 3 case in [14].

Then what about the exact formulae in the rank modulo 5 and 7 cases, where Ramanujan's congruences appear? Bringmann [17] proved the general asymptotics for all odd moduli, while the Kloosterman type sums are hard to interpret as Kloosterman sums. Thanks to the theory of vector-valued Maass forms from [8] and the explicit transformation laws by Garvan [18], the author finds the interpretation as vector-valued Kloosterman sums. Combining with some generalization of [16], the author finally provides the proof for the exact formula of rank modulo primes $p \geq 5$. The author also has a striking observation between the interesting cases $p = 5, 7$, where the Kloosterman sums become identically zero (or become equal for those defined on different cusp pairs). After a long study of the cases depending on congruence properties of the Dedekind sums, the author proves this cancellation property and provides a new proof for the Dyson's conjecture $5N(a, 5; 5n + 4) = p(5n + 4)$ and $7N(a, 7; 7n + 5) = p(7n + 5)$ which implies Ramanujan's congruences.

*To my love, Biyu.*

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海客談瀛洲，煙濤微茫信難求；

越人語天姥，雲霞明滅或可覩。

(Of fairy isles seafarers speak, 'Mid dimming mist and surging waves, so hard to seek'; Of Skyland Southerners are proud, perceivable through fleeting or dispersing cloud. - Translated by Prof. Yuanchong Xu.)

# Table of contents

# Chapter 1

# Introduction

In this thesis, we collect the author's results on estimates for sums of Kloosterman sums and their applications to the theory of integer partitions. These estimations generalize the work of Goldfeld and Sarnak [16], Sarnak and Tsimerman [19], as well as the work of Ahlgren, Andersen and Dunn [11], [12], [20]. Thanks to the theory for ranks of integer partitions developed by Bringmann, Ono and Garvan [7], [8], [17], [18], [21], the author is able to apply the certain estimates, especially the uniform bounds for sums of Kloosterman sums, to generalize Rademacher's exact formula for ranks of partitions modulo primes.

The first paper [14] for the mixed-sign case of Theorem 1.7 has been published in *Forum Mathematicum* and we record the proof mainly in Chapter 4. The second one for the same-sign case of Theorem 1.7 was submitted for publication and we record the proof in Chapter 5. The author's work on the proof of Theorem 1.14 and Theorem 1.15 has not been submitted yet, but we record these results in Chapter 7 and Chapter 8 in this thesis.

## 1.1 Standard Kloosterman sums

For a positive integer $c$, the standard Kloosterman sum

$$S(m,n,c) := \sum_{d \pmod{c)^*}} e\left(\frac{m\overline{d}+nd}{c}\right), \quad e(z) := e^{2\pi i z}, \quad \overline{d}d \equiv 1 \pmod{c}$$

has a trivial bound $c$ and a well-known Weil bound

$$|S(m,n,c)| \leq \sigma_0(c)(m,n,c)^{\frac{1}{2}}c^{\frac{1}{2}}, \tag{1.1}$$

where $\sigma_k(\ell) = \sum_{d|\ell} d^k$ is the divisor function. The Weil bound implies a square-root cancellation for estimating

$$\sum_{c\leq x} \frac{S(m,n,c)}{c} \ll \sigma_0((m,n))x^{\frac{1}{2}}\log x. \tag{1.2}$$

In the 1960s, Linnik [22] and Selberg [23] pointed out the connection between such sums and modular forms. They conjectured that there should be a full cancellation, which was reformulated by Sarnak and Tsimerman in [19] as

$$\sum_{c\le x}\frac{S(m,n,c)}{c}\ll_\varepsilon |mnx|^\varepsilon.$$

Kuznetsov [24] applied his famous trace formula which resulted in the bound

$$\sum_{c\le x}\frac{S(m,n,c)}{c}\ll_{m,n} x^{\frac16}(\log x)^{\frac13}.$$

Sarnak and Tsimerman [19] obtained a bound which is uniform in $m$ and $n$ for $mn>0$:

$$\sum_{c\le x}\frac{S(m,n,c)}{c}\ll_\varepsilon \left(x^{\frac16}+(mn)^{\frac16}+(m+n)^{\frac18}(mn)^{\frac{\theta}{2}}\right)(mnx)^\varepsilon, \tag{1.3}$$

where $\theta$ is an admissible exponent towards the Ramanujan-Petersson conjecture for $\mathrm{GL}_2/\mathbb{Q}$. One may take $\theta=\frac{7}{64}$ by the work of Kim and Sarnak [25].

## 1.2 Multiplier systems and general Kloosterman sums

Denote the modular group $\mathrm{SL}_2(\mathbb{Z}):=\{\gamma\in\mathrm{M}_2(\mathbb{Z}):\det\gamma=1\}$, where $\mathrm{M}_2(\mathbb{Z})$ is the set of 2 by 2 matrices with integer entries. Each $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$ acts on the upper-half complex plane

$$\mathbb{H}:=\{z\in\mathbb{C}:\quad z=x+iy,\quad x,y\in\mathbb{R},\quad y=\operatorname{Im}z>0\}$$

as a Mobiüs transformation $z\to\gamma z:=\frac{az+b}{cz+d}$. This operation satisfies $\gamma_1(\gamma_2 z)=(\gamma_1\gamma_2)z$. When $c\neq0$, this definition can be extended to $\mathbb{H}\cup\{\infty\}\cup\mathbb{Q}$ by defining

$$\gamma(\tfrac{-d}{c})=\infty\quad\text{and}\quad\gamma\infty=\frac{a}{c};$$

when $c=0$, we define $\gamma\infty=\infty$.

For any subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ of finite index, the quotient topological space $\Gamma\setminus\mathbb{H}$ is a Hausdorff space. After adding the set of points $\mathbb{Q}\cup\{\infty\}$, $\Gamma\setminus\mathbb{H}$ is compactified. We define the cusps of $\Gamma\setminus\mathbb{H}$, or simply the cusps of $\Gamma$, as the equivalence classes of $\mathbb{Q}\cup\{\infty\}$ under the action of $\Gamma$. There are several important subgroups of $\mathrm{SL}_2(\mathbb{Z})$: let $N\ge1$ be an integer, we define

$$\Gamma_0(N):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}):\ c\equiv0\ (\mathrm{mod}\ N)\right\}, \tag{1.4}$$

$$\Gamma_1(N):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}):\ a,d\equiv1\ (\mathrm{mod}\ N),\ c\equiv0\ (\mathrm{mod}\ N)\right\}, \tag{1.5}$$

$$\Gamma(N):=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}):\ a,d\equiv1\ (\mathrm{mod}\ N),\ b,c\equiv0\ (\mathrm{mod}\ N)\right\}. \tag{1.6}$$

Fixing the argument $(-\pi,\pi]$, for any $\gamma\in\mathrm{SL}_2(\mathbb{R})$ and $z=x+iy\in\mathbb{H}$, we define the automorphic factor

$$j(\gamma,z):=\frac{cz+d}{|cz+d|}=e^{i\arg(cz+d)} \tag{1.7}$$

and the weight $k$ slash operator

$$(f|_k\gamma)(z) := j(\gamma, z)^{-k} f(\gamma z) \tag{1.8}$$

for $k \in \mathbb{R}$.

**Definition 1.1.** *We say that $\nu : \Gamma \to \mathbb{C}^\times$ is a multiplier system of weight $k$ if*

*(i)* $|\nu| = 1$,

*(ii)* $\nu(-I) = e^{-\pi i k}$, *and*

*(iii)* $\nu(\gamma_1\gamma_2) = w_k(\gamma_1, \gamma_2)\nu(\gamma_1)\nu(\gamma_2)$ *for all* $\gamma_1, \gamma_2 \in \Gamma$, *where*

$$w_k(\gamma_1, \gamma_2) := j(\gamma_2, z)^k j(\gamma_1, \gamma_2 z)^k j(\gamma_1\gamma_2, z)^{-k}.$$

If $\nu$ is a multiplier system of weight $k$, then it is also a multiplier system of weight $k'$ for any $k' \equiv k \pmod 2$, and its conjugate $\overline{\nu}$ is a multiplier system of weight $-k$. One can check the basic properties that

$$\nu(\gamma)\nu(\gamma^{-1}) = 1, \quad \nu(\gamma(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix})) = \nu(\gamma)\nu((\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}))^b. \tag{1.9}$$

For any cusp $\mathfrak{a}$ of a congruence subgroup $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ where $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}) \in \Gamma$, let $\Gamma_\mathfrak{a}$ denote its stabilizer in $\Gamma$. For example, $\Gamma_\infty = \{\pm(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}) : b \in \mathbb{Z}\}$. Let $\sigma_\mathfrak{a} \in \mathrm{SL}_2(\mathbb{R})$ denote a scaling matrix of $\mathfrak{a}$, which means $\sigma_\mathfrak{a}$ satisfies

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty. \tag{1.10}$$

We define $\alpha_{\nu,\mathfrak{a}} \in [0, 1)$ by the condition

$$\nu\left(\sigma_\mathfrak{a}(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})\sigma_\mathfrak{a}^{-1}\right) = e(-\alpha_{\nu,\mathfrak{a}}). \tag{1.11}$$

The cusp $\mathfrak{a}$ is called singular if $\alpha_{\nu,\mathfrak{a}} = 0$. When $\mathfrak{a} = \infty$ we drop the subscript and denote $\alpha_\nu := \alpha_{\nu,\infty}$. For $n \in \mathbb{Z}$, define $n_\mathfrak{a} := n - \alpha_{\nu,\mathfrak{a}}$ and $n_\infty = \tilde{n} := n - \alpha_\nu$.

The Kloosterman sums for the cusp pair $(\infty, \infty)$ with respect to $\nu$ are given by

$$S(m, n, c, \nu) := \sum_{\substack{0 \le a, d < c \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma}} \overline{\nu}(\gamma) e\left(\frac{\tilde{m}a + \tilde{n}d}{c}\right) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \overline{\nu}(\gamma) e\left(\frac{\tilde{m}a + \tilde{n}d}{c}\right). \tag{1.12}$$

They satisfy the relationships

$$\overline{S(m, n, c, \nu)} = \begin{cases} S(1 - m, 1 - n, c, \overline{\nu}) & \text{if } \alpha_\nu > 0, \\ S(-m, -n, c, \overline{\nu}) & \text{if } \alpha_\nu = 0, \end{cases} \tag{1.13}$$

because

$$n_{\overline{\nu}} = \begin{cases} -(1 - n)_\nu & \text{if } \alpha_\nu > 0, \\ n & \text{if } \alpha_\nu = 0. \end{cases} \tag{1.14}$$

There are two fundamental multiplier systems of weight $\frac{1}{2}$. The theta-multiplier $\nu_\theta$ on $\Gamma_0(4)$ is given by

$$\theta(\gamma z) = \nu_\theta(\gamma)\sqrt{cz + d}\,\theta(z), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4) \tag{1.15}$$

where

$$\theta(z) := \sum_{n\in\mathbb{Z}} e(n^2 z), \quad \nu_\theta(\gamma) = \left(\frac{c}{d}\right)\varepsilon_d^{-1}, \quad \varepsilon_d = \begin{cases} 1 & d \equiv 1 \pmod 4, \\ i & d \equiv 3 \pmod 4, \end{cases}$$

and $\left(\frac{\cdot}{\cdot}\right)$ is the extended Kronecker symbol. The eta-multiplier $\nu_\eta$ on $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\eta(\gamma z) = \nu_\eta(\gamma)\sqrt{cz+d}\,\eta(z), \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \tag{1.16}$$

where

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n), \quad q = e(z). \tag{1.17}$$

Let $((x)) := x - \lfloor x \rfloor - \frac12$ when $x \in \mathbb{R}\setminus\mathbb{Z}$ and $((x)) := 0$ when $x \in \mathbb{Z}$. We have the explicit formula [26, (74.11), (74.12)]

$$\nu_\eta(\gamma) = e\left(-\frac18\right) e^{-\pi i s(d,c)} e\left(\frac{a+d}{24c}\right), \quad s(d,c) := \sum_{r \pmod c} \left(\left(\frac{r}{c}\right)\right)\left(\left(\frac{dr}{c}\right)\right), \tag{1.18}$$

for all $c \in \mathbb{Z}\setminus\{0\}$ and $\nu_\eta\left(\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\right) = e\left(\frac{b}{24}\right)$. Another formula [27] for $c > 0$ is

$$\nu_\eta(\gamma) = \begin{cases} \left(\frac{d}{c}\right) e\left\{\frac{1}{24}\Big((a+d)c - bd(c^2-1) - 3c\Big)\right\} & \text{if } c \text{ is odd}, \\ \left(\frac{c}{d}\right) e\left\{\frac{1}{24}\Big((a+d)c - bd(c^2-1) + 3d - 3 - 3cd\Big)\right\} & \text{if } c \text{ is even}. \end{cases} \tag{1.19}$$

The properties $\nu_\eta(-\gamma) = i\nu_\eta(\gamma)$ when $c > 0$ and $e(\frac{1-d}{8}) = (\frac{2}{d})\varepsilon_d$ for odd $d$ are convenient.

## 1.3 Estimates for sums of Kloosterman sums

There is a famous result by Goldfeld and Sarnak [16] estimating the sums of general Kloosterman sums. Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Let $k \in \mathbb{R}$ and $\nu$ be a weight $k$ multiplier system on $\Gamma$. Define

$$\beta := \limsup_{c\to\infty} \frac{\log|S(m,n,c,\nu)|}{\log c}. \tag{1.20}$$

For $m, n \in \mathbb{Z}$, we define the Kloosterman-Selberg zeta function as

$$Z_{m,n,\nu}(s) := \sum_{c=1}^{\infty} \frac{S(m,n,c,\nu)}{c^{2s}}. \tag{1.21}$$

By the definition of $\beta$, one can see that $Z_{m,n,\nu}(\cdot)$ is defined and holomorphic on $\operatorname{Re} s > \frac{\beta+1}{2}$. Goldfeld and Sarnak proved the following theorem.

**Theorem 1.2** ([16, Theorem 1]). *The function $Z_{m,n,\nu}(s)$ is meromorphic in $\operatorname{Re} s > \frac12$ with at most a finite number of simple poles in $(\frac12, 1)$.*

Based on the growth condition of $Z_{m,n,\nu}(s)$, Goldfeld and Sarnak obtained the following estimate:

**Theorem 1.3** ([16, Theorem 2]). *For any $\varepsilon > 0$,*

$$\sum_{c\le x} \frac{S(m,n,c,\nu)}{c} = \sum_{s_j \in (\frac12, 1)} \tau_j(m,n) \frac{x^{2s_j-1}}{2s_j - 1} + O_{m,n,k,\Gamma,\nu,\varepsilon}\left(x^{\frac{\beta}{3}+\varepsilon}\right).$$

*Here the sum runs over the simple poles of $Z_{m,n,\nu}(s)$ in $(\frac{1}{2}, 1)$ and $\tau_j(m,n)$ depends on $m, n, \nu$, and $\Gamma$.*

We will show the formula for $\tau_j(m,n)$ in Theorem 1.7 but not repeat it here. The above bound does not show the dependence on $m$ and $n$, while the methods in [16] guaranteed a polynomial growth for them. We will leave this discussion until Chapter 3. The uniform bounds for sums of general Kloosterman sums, like (1.3), has been obtained by Ahlgren and Andersen in special cases:

**Theorem 1.4** ([11, Theorem 1.3, Theorem 9.1])**.** *For $m > 0$ and $n < 0$ we have*

$$\sum_{c \le x} \frac{S(m,n,c,\nu_\eta)}{c} \ll_\varepsilon \left(x^{\frac{1}{6}} + |mn|^{\frac{1}{4}}\right) |mn|^\varepsilon \log x.$$

*Moreover, for $n < 0$ and $0 < \delta < \frac{1}{2}$, we have*

$$\sum_{c \le X} \frac{S(1,n,c,\nu_\eta)}{c} \ll_{\delta,\varepsilon} |n|^{\frac{13}{56}+\varepsilon} X^{\frac{3}{4}\delta} + \left(|n|^{\frac{41}{168}+\varepsilon} + X^{\frac{1}{2}-\delta}\right) \log X.$$

Let $\psi$ be the conjugate of the weight $\frac{3}{2}$ multiplier system of $\eta(z)^5/\eta(2z)^2$ on $\Gamma_0(2)$ (hence $\psi$ is a weight $\frac{1}{2}$ multiplier system on $\Gamma_0(2)$), then $\psi$ is exactly the multiplier system defined at [12, (3.4)]. Ahlgren and Dunn proved:

**Theorem 1.5** ([12, Theorem 7.1])**.** *Suppose that $24n - 1$ is positive and squarefree and that $0 < \delta < \frac{1}{2}$. For $X \ge 1$ and $\varepsilon > 0$ we have*

$$\sum_{2|c \le x} \frac{S(0,n,c,\psi)}{c} \ll_{\delta,\varepsilon} |n|^{\frac{13}{56}+\varepsilon} X^{\frac{3}{4}\delta} + \left(|n|^{\frac{143}{588}+\varepsilon} + X^{\frac{1}{2}-\delta}\right) X^\varepsilon.$$

The author is able to generalize these results to a wide class of half-integral weight multiplier systems. Besides the uniform bound, the author also recovers the $\tau_j$ terms in Goldfeld and Sarnak's result (Theorem 1.3) via the trace formula, which is a different method from the original paper.

**Definition 1.6** (Definition 1.1 in [14], [15])**.** *Let $(k, \nu') = (\frac{1}{2}, (\frac{|D|}{\cdot})\nu_\theta)$ or $(-\frac{1}{2}, (\frac{|D|}{\cdot})\overline{\nu_\theta})$ where $D$ is some even fundamental discriminant and $\nu_\theta$ is the multiplier for the theta function. We say that a weight $k$ multiplier $\nu$ on $\Gamma = \Gamma_0(N)$ is <u>admissible</u> if it satisfies the following two conditions:*

*(1) Level lifting: there exist positive integers $B$ and $M$ such that the map $\mathscr{L} : (\mathscr{L}f)(z) = f(Bz)$ gives:*

  *(i) an injection from weight $k$ automorphic eigenforms of the hyperbolic Laplacian $\Delta_k$ on $(\Gamma_0(N), \nu)$ to those on $(\Gamma_0(M), \nu')$ and keeps the eigenvalue;*

  *(ii) an injection from weight $k$ holomorphic cusp forms on $(\Gamma_0(N), \nu)$ to weight $k$ holomorphic cusp forms on $(\Gamma_0(M), \nu')$.*

  *Here $M$ is a multiple of 4 and $M$ depends on $B$.*

*(2) Average Weil bound: for $x > y > 0$ and $x - y \gg x^{\frac{2}{3}}$, we have*

$$\sum_{N|c \in [y,x]} \frac{|S(m,n,c,\nu)|}{c} \ll_{\nu,\varepsilon} (\sqrt{x} - \sqrt{y}) |\tilde{m}\tilde{n}x|^\varepsilon.$$

*Remark.* The exponent $\frac{2}{3} = 1 - \delta$ comes from a parameter $\delta$ in the proof which is finally chosen to be $\frac{1}{3}$. An individual Weil-type bound on $S(m, n, c, \nu)$ can imply the average bound specified in condition (2), but our result only needs this weaker requirement.

The author proved the following theorem in two papers and states the results together here. The difference is that [14] is for the case $\tilde{m}\tilde{n} < 0$ while [15] is for the case $\tilde{m}\tilde{n} > 0$. Although the two cases and conclusions look similar, there are significant differences in the proofs.

**Theorem 1.7** ([14, Theorem 1.4], [15, Theorem 1.3])**.** *Suppose $\tilde{m}\tilde{n} \neq 0$ and $\nu$ is a weight $k = \pm\frac{1}{2}$ admissible multiplier on $\Gamma_0(N)$. We have*

$$\sum_{N|c\leq X} \frac{S(m,n,c,\nu)}{c} = \sum_{s_j \in (\frac{1}{2},\frac{3}{4})} \tau_j(m,n) \frac{X^{2s_j-1}}{2s_j-1} + O_{\nu,\varepsilon}\left(\left(A_u(m,n) + X^{\frac{1}{6}}\right)(\tilde{m}\tilde{n}X)^{\varepsilon}\right), \tag{1.22}$$

*where for $B$ and $M$ in Definition 1.6, we factor $B\tilde{\ell} = t_\ell u_\ell^2 w_\ell^2$ with $t_\ell$ square-free, $u_\ell | M^\infty$ positive and $(w_\ell, M) = 1$ for $\ell \in \{m, n\}$. Here $\tau_j(m,n)$ are the coefficients in [16] (as corrected by [28, Proposition 7]):*

$$\tau_j(m,n) = 2i^k \overline{\rho_j(m)} \rho_j(n) \pi^{1-2s_j} (4\tilde{m}\tilde{n})^{1-s_j} \frac{\Gamma(s_j + \operatorname{sgn}\tilde{n} \cdot \frac{k}{2})\Gamma(2s_j-1)}{\Gamma(s_j - \frac{k}{2})},$$

*and*

$$\begin{aligned} A_u(m,n) &:= \left(\tilde{m}^{\frac{131}{294}} + u_m\right)^{\frac{1}{8}} \left(\tilde{n}^{\frac{131}{294}} + u_n\right)^{\frac{1}{8}} (\tilde{m}\tilde{n})^{\frac{3}{16}} \\ &\ll (\tilde{m}\tilde{n})^{\frac{143}{588}} + \tilde{m}^{\frac{143}{588}} \tilde{n}^{\frac{3}{16}} u_n^{\frac{1}{8}} + \tilde{m}^{\frac{3}{16}} \tilde{n}^{\frac{143}{588}} u_m^{\frac{1}{8}} + (\tilde{m}\tilde{n})^{\frac{3}{16}} (u_m u_n)^{\frac{1}{8}}. \end{aligned}$$

As a corollary or a simpler version of the above theorem, we have

**Corollary 1.8.** *With the same setting and notations as Theorem 1.7, we suppose $B\tilde{\ell}$ is square-free or coprime to $N$ for $\ell \in \{m, n\}$, then*

$$\sum_{N|c\leq X} \frac{S(m,n,c,\nu)}{c} = \sum_{s_j \in (\frac{1}{2},\frac{3}{4})} \tau_j(m,n) \frac{X^{2s_j-1}}{2s_j-1} + O_{\nu,\varepsilon}\left(\left(|\tilde{m}\tilde{n}|^{\frac{143}{588}} + X^{\frac{1}{6}}\right)|\tilde{m}\tilde{n}X|^{\varepsilon}\right). \tag{1.23}$$

*Remark.* We have the following notes for the theorem and corollary above:

- The notation $u|M^\infty$ means $u|M^C$ for some positive integer $C$.
- When $u_m$ and $u_n$ are both $O_{N,\nu}(1)$, we have $A_u(m,n) \ll_{N,\nu} |\tilde{m}\tilde{n}|^{\frac{143}{588}}$.
- In general, $A_u(m,n) \ll_{N,\nu} (\tilde{m}\tilde{n})^{\frac{1}{4}}$.
- The theorem also applies to the case $\tilde{m} < 0$ and $\tilde{n} < 0$ because of (1.13) by conjugation.
- When $r_j = \frac{i}{4}$, we have $\tau_j(m,n) = 0$ unless $\operatorname{sgn}\tilde{m}$, $\operatorname{sgn}\tilde{n}$, and $\operatorname{sgn} k$ are all the same (see (2.13) and (2.14)).

We modify our estimate to get the following bound suitable for the applications in Chapter 6. Recall that $\alpha_\nu \in [0,1)$ is defined as $\nu(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)) = e(-\alpha_\nu)$.

**Theorem 1.9** ([14, Theorem 1.6], [15, Theorem 1.5]). *With the same setting as Theorem 1.7, we suppose $B\tilde{\ell}$ is square-free or coprime to $N$ for $\ell \in \{m, n\}$. Then for $\beta = \frac{1}{2}$ or $\frac{3}{2}$ and $\alpha > 0$, when $\tau_j(m, n) = 0$ for $r_j = \frac{i}{4}$, we have*

$$\sum_{N|c>\alpha\sqrt{|\tilde{m}\tilde{n}|}} \frac{S(m,n,c,\nu)}{c} \mathscr{M}_\beta\left(\frac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right) \ll_{\alpha,\nu,\varepsilon} |\tilde{m}\tilde{n}|^{\frac{143}{588}+\varepsilon}, \tag{1.24}$$

*where $\mathscr{M}_\beta$ is the Bessel function $I_\beta$ or $J_\beta$.*

*Remark.* We prove Theorem 1.9 in the end of Chapter 5. The best bound, in the particular case $S(1, n, c, \nu_\eta)$ with the eta-multiplier, was recently given by Andersen and Wu [13, (2.10)]: for $n > 0$,

$$\sum_{c\le x} \frac{S(1,1-n,c,\nu_\eta)}{c} \ll_\varepsilon \left(x^{\frac{1}{6}} + |d|^{\frac{2}{9}} w^{\frac{1}{3}}\right)(nx)^\varepsilon \tag{1.25}$$

where $d$ and $w$ are given by $1 - 24n =: dw^2$ such that $d \equiv 1 \pmod{24}$ is a negative fundamental discriminant. They proved this stronger bound by applying a hybrid subconvexity bound of twisted $L$-functions which generalizes Young's result in [29].

## 1.4 Ranks of partitions and Rademacher-type exact formulas

From now on, we denote $p(n)$ as the partition function, which is the number of ways to write the natural number $n$ as a sum of a non-increasing sequence of positive integers. For example, we have $p(3) = 3$ ($3$, $2+1$ and $1+1+1$), $p(4) = 5$ ($4$, $3+1$, $2+2$, $2+1+1$ and $1+1+1+1$), and $p(100) = 190\,569\,292$. In 1918, Hardy and Ramanujan [4] proved the asymptotics for $p(n)$:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right).$$

Later, in 1938, Rademacher [5] proved the exact formula of $p(n)$. If we define

$$A_c(n) := \frac{1}{2}\sqrt{\frac{c}{12}} \sum_{\substack{x \pmod{24c} \\ x^2 \equiv -24n+1 \pmod{24c}}} \chi_{12}(x) e\left(\frac{x}{12c}\right), \tag{1.26}$$

where $\chi_{12}$ is the Dirichlet character $(\frac{12}{\cdot})$ modulo 12, $e(z) := e^{2\pi i z}$, and the sum runs over the residue classes modulo $24c$, then Rademacher's exact formula [5, (1.8)] can be written as [11, (1.2), (1.3)]

$$\begin{aligned} p(n) &= \frac{1}{\pi\sqrt{2}} \sum_{c=1}^\infty A_c(n)\sqrt{c}\, \frac{d}{dn}\left(\frac{\sinh\left(\frac{\pi}{c}\sqrt{\frac{2}{3}(n-\frac{1}{24})}\right)}{\sqrt{n-\frac{1}{24}}}\right) \\ &= \frac{2\pi}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^\infty \frac{A_c(n)}{c} I_{\frac{3}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right) \\ &= \frac{2\pi e(-\frac{1}{8})}{(24n-1)^{\frac{3}{4}}} \sum_{c=1}^\infty \frac{S(1,1-n,c,\nu_\eta)}{c} I_{\frac{3}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right). \end{aligned} \tag{1.27}$$

Ramanujan also obtained the famous congruence properties of $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}. \tag{1.28}$$

In 1944, Dyson [6] defined the rank of a partition to strikingly interpret the above congruences. Suppose $\Lambda = \{\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_\kappa\}$ is a partition of $n$, i.e. $\sum_{j=1}^{\kappa} \Lambda_j = n$. Let

$$\operatorname{rank}(\Lambda) := \Lambda_1 - \kappa$$

define the rank of this partition, and let the quantities $N(m,n)$ and $N(a,b;n)$ be defined by

$$N(m,n) := \#\{\Lambda \text{ is a partition of } n : \operatorname{rank}\Lambda = m\} \tag{1.29}$$

and

$$N(a,b;n) := \#\{\Lambda \text{ is a partition of } n : \operatorname{rank}\Lambda \equiv a \pmod b\}. \tag{1.30}$$

Let $q = \exp(2\pi i z) = e(z)$ for $z \in \mathbb{H}$ (the upper-half complex plane) and $w$ be a root of unity. It is well known (e.g. [8, (1.4)]) that the generating function of $N(m,n)$ can be written as

$$\mathcal{R}(w;q) := 1 + \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n(w^{-1}q;q)_n}, \tag{1.31}$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j)$. For example, $\mathcal{R}(1;q) = 1 + \sum_{n=1}^{\infty} p(n)q^n$ is the generating function for partitions. For integers $b > a > 0$, denote $A(\frac{a}{b};n)$ as the Fourier coefficient of $\mathcal{R}(\zeta_b^a;q)$:

$$\mathcal{R}(\zeta_b^a;q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{b};n\right) q^n$$

where $\zeta_b = \exp(\frac{2\pi i}{b})$ is a $b$-th root of unity. The following identity is easy to get but helpful in understanding the relation between $A(\frac{a}{b};n)$ and $N(a,b;n)$:

$$bN(a,b;n) = p(n) + \sum_{j=1}^{b-1} \zeta_b^{-aj} A\left(\frac{j}{b};n\right). \tag{1.32}$$

It is not hard to show that $A(\frac{j}{b};n) \in \mathbb{R}$ for $1 \leq j \leq b-1$ because $N(a,b;n) = N(b-a,b;n)$, $A(\frac{j}{b};n) = A(\frac{b-j}{b};n)$ and $\zeta_b^{-aj} + \zeta_b^{-a(b-j)} \in \mathbb{R}$.

The function $\mathcal{R}(w;q)$ has many beautiful connections and properties. When $w = -1$, it is known that $N(0,2;n) - N(1,2;n) = A(\frac{1}{2};n)$ is the Fourier coefficient of Ramanujan's third order mock theta function $f(q)$. We know that the Hardy-Ramanujan asymptotic

$$p(n) \sim \frac{1}{4n\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}n}}$$

was perfected by Rademacher's exact formula (1.27). Similarly, Dragonette [9] and Andrews [10] improved the asymptotic formula of $A(\frac{1}{2};n)$ which was conjectured by Ramanujan. The exact formula for $A(\frac{1}{2};n)$ was later proven by Bringmann and Ono:

**Theorem 1.10** ([7, Theorem 1.1])**.** *The Andrews-Dragonette conjecture is true:*

$$A\left(\frac{1}{2};n\right)=\frac{\pi}{(24n-1)^{\frac14}}\sum_{k=1}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}A_{2k}(n-\frac{k(1+(-1)^k)}{4})}{k}I_{\frac12}\left(\frac{\pi\sqrt{24n-1}}{12k}\right). \tag{1.33}$$

*Remark.* Recall the weight $\frac12$ multiplier $\psi$ on $\Gamma_0(2)$ defined before Theorem 1.5 (see [12, (3.4), Lemma 3.2]). By [12, (3.5)] and [12, Lemma 3.1], we have

$$(-1)^{\lfloor\frac{c+1}{2}\rfloor}A_{2c}\left(n-\frac{c(1+(-1)c)}{4}\right)=e(\tfrac18)\overline{S(0,n,2,\psi)},$$

so we can rewrite the exact formula (1.33) of $A(\frac12;n)$ as

$$A\left(\frac{1}{2};n\right)=\frac{2\pi\, e(-\frac18)}{(24n-1)^{\frac14}}\sum_{2|c>0}\frac{S(0,n,c,\psi)}{c}I_{\frac12}\left(\frac{\pi\sqrt{24n-1}}{6c}\right). \tag{1.34}$$

The author [14] provided a detailed proof of $A(\frac13;n)=A(\frac23;n)$, which is the Fourier coefficient of $\mathcal{R}(\zeta_3;q)=\mathcal{R}(\zeta_3^2;q)$.

**Theorem 1.11** ([14, Theorem 2.2])**.** *We have*

$$A\left(\frac{1}{3};n\right)=A\left(\frac{2}{3};n\right)=\frac{2\pi\, e(-\frac18)}{(24n-1)^{\frac14}}\sum_{3|c>0}\frac{S(0,n,c,\,(\frac{\cdot}{3})\overline{\nu_\eta})}{c}I_{\frac12}\left(\frac{\pi\sqrt{24n-1}}{6c}\right). \tag{1.35}$$

Let $R_j(n,x)$ be the tail sum on $n>x$ of the above exact formulas for $A(\frac1j;n)$: $j=1$ for (1.27), $j=2$ for (1.33) (see (1.34)) and $j=3$ for (1.35). For example,

$$R_3(n,x)=\frac{2\pi\, e(-\frac18)}{(24n-1)^{\frac14}}\sum_{3|c>x}\frac{S(0,n,c,\,(\frac{\cdot}{3})\overline{\nu_\eta})}{c}I_{\frac12}\left(\frac{\pi\sqrt{24n-1}}{6c}\right). \tag{1.36}$$

Each multiplier system $\nu$ in these sums satisfies $\alpha_\nu=\frac{1}{24}$ or $\frac{23}{24}$ with $B=24$ and $M|576$ in Definition 1.6. Clearly both $24n-1$ and $24n-23$ are always coprime with $M$. The admissibility of each multiplier is proved in [11], [12] and Proposition 4.1, respectively. Now we can apply Theorem 1.9 to get a power-saving with exponent $\frac14-\frac{143}{588}=\frac{1}{147}$ less:

**Theorem 1.12.** *For $\alpha>0$ we have*

$$R_j(n,\alpha\sqrt{n})\ll_{\alpha,\varepsilon}\begin{cases} n^{-\frac12-\frac{1}{147}+\varepsilon}, & j=1;\\ n^{-\frac{1}{147}+\varepsilon}, & j=2,3.\end{cases}$$

*Remark.* The previous results on the growth rates for $R_j(n,\alpha\sqrt{n})$ are $n^{-\frac12+\varepsilon}$ for $j=1$ and $n^\varepsilon$ for $j=2,3$. These results come from careful applications of the circle method. We will discuss these milestones in Chapter 6.

When $j=1,2$, this improves [12, Theorem 1.4, Theorem 1.1] by removing the square-free requirement. Recently Andersen and Wu [13, Theorem 1.1] proved a stronger bound when $j=1$ based on their estimate (1.25):

$$R_1(n,\alpha\sqrt{n})\ll_{\alpha,\varepsilon} n^{-\frac12-\frac{1}{36}+\varepsilon}.$$

Another new contribution in this thesis is to extend to the case $j = 3$.

In 2009, Bringmann [17] used the circle method to find the asymptotics of $A(\frac{\ell}{u}; n)$ for general odd $u$. Let $s(d,c)$ be the Dedekind sum defined in (1.18) and $\omega_{d,c} := \exp(\pi i s(d,c))$. When $(d,c) = 1$, define $d'_c$ by $dd'_c \equiv -1 \pmod{c}$ if $c$ is odd and $dd'_c \equiv -1 \pmod{2c}$ if $c$ is even. Denote

$$g_{(h)} := \frac{g}{\gcd(g,h)}$$

for non-zero integers $g$ and $h$. If $u|c$, define

$$B_{\ell,u,c}(n,m) := (-1)^{\ell c+1} \sum_{d \pmod{c}^*} \frac{\sin(\frac{\pi \ell}{u})\, \omega_{d,c}}{\sin(\frac{\pi \ell d'_c}{u}) \exp(\frac{3\pi i c_{(u)} d'_c \ell^2}{u})} e\left(\frac{md'_c + nd}{c}\right). \tag{1.37}$$

When $u \nmid a$ and $1 \le \ell < u_{(a)}$, let $0 \le [a_{(u)}\ell] < u_{(a)}$ be defined by $[a_{(u)}\ell] \equiv a_{(u)}\ell \pmod{u_{(a)}}$. Define

$$D_{\ell,u,a}(n,m) := (-1)^{a\ell + [a_{(u)}\ell]} \sum_{b \pmod{a}^*} \omega_{b,a} e\left(\frac{mb'_a + nb}{a}\right). \tag{1.38}$$

When $u \nmid a$, define

$$\delta_{\ell,u,a,r} := \begin{cases} -(\frac{1}{2}+r)\frac{[a_{(u)}\ell]}{u_{(a)}} + \frac{3}{2}\left(\frac{[a_{(u)}\ell]}{u_{(a)}}\right)^2 + \frac{1}{24}, & \text{if } 0 < \frac{[a_{(u)}\ell]}{u_{(a)}} < \frac{1}{6}, \\ -\frac{5[a_{(u)}\ell]}{2u_{(a)}} + \frac{3}{2}\left(\frac{[a_{(u)}\ell]}{u_{(a)}}\right)^2 + \frac{25}{24} - r + \frac{r[a_{(u)}\ell]}{u_{(a)}}, & \text{if } \frac{5}{6} < \frac{[a_{(u)}\ell]}{u_{(a)}} < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.39}$$

and when $0 < \frac{[a_{(u)}\ell]}{u_{(a)}} < \frac{1}{6}$ or $\frac{5}{6} < \frac{[a_{(u)}\ell]}{u_{(a)}} < 1$, define

$$m_{\ell,u,a,r} := \begin{cases} \frac{1}{2u_{(a)}^2}\Big(-3\left(a_{(u)}\ell - [a_{(u)}\ell]\right)^2 - u_{(a)}(1+2r)\left(a_{(u)}\ell - [a_{(u)}\ell]\right)\Big), & \text{if } 0 < \frac{[a_{(u)}\ell]}{u_{(a)}} < \frac{1}{6}, \\ \frac{1}{2u_{(a)}^2}\Big(-3\left(a_{(u)}\ell - [a_{(u)}\ell]\right)^2 + u_{(a)}(2r-5)\left(a_{(u)}\ell - [a_{(u)}\ell]\right) + 2u_{(a)}^2(r-1)\Big), & \text{if } \frac{5}{6} < \frac{[a_{(u)}\ell]}{u_{(a)}} < 1. \end{cases} \tag{1.40}$$

By [17, bottom of p. 3485], or directly by $u_{(a)} \Big| \left(a_{(u)}\ell - [a_{(u)}\ell]\right)$, we can see $m_{\ell,u,a,r} \in \mathbb{Z}$ always.

Bringmann proved:

**Theorem 1.13** ([17, Theorem 1.1]). *If $1 \le \ell < u$ are coprime integers and $u$ is odd, then for positive integers $n$ we have*

$$\begin{aligned} A\left(\frac{\ell}{u}; n\right) &= \frac{4\sqrt{3}\, i}{\sqrt{24n-1}} \sum_{c:\ u|c \le \sqrt{n}} \frac{B_{\ell,u,c}(-n,0)}{\sqrt{c}} \sinh\left(\frac{\pi\sqrt{24n-1}}{6c}\right) \\ &+ \frac{8\sqrt{3}\sin(\frac{\pi\ell}{u})}{\sqrt{24n-1}} \sum_{r\ge 0} \sum_{\substack{a \le \sqrt{n}: \\ u \nmid a, \\ \delta_{\ell,u,a,r} > 0}} \frac{D_{\ell,u,a}(-n, m_{\ell,u,a,r})}{\sqrt{a}} \sinh\left(\frac{\pi\sqrt{2\delta_{\ell,u,a,r}(24n-1)}}{a\sqrt{3}}\right) + O_{u,\varepsilon}(n^{\varepsilon}). \end{aligned} \tag{1.41}$$

Note that the sum of $r \ge 0$ in the second line is a finite sum because when $v$ is fixed and $r$ is large enough, $\delta_{\ell,v,a,r}$ will be always negative. Here we have modified the notation in Bringmann's paper for convenience in this thesis. Bringmann and Ono [21] claimed that the above sum, when summing up to infinity, should be the exact formula for $A(\frac{\ell}{u};n)$.

When $u=p$ is a prime number, the author proves that their statement is true. We also explain $B_{\ell,p,c}(-n,0)$ and $D_{\ell,p,a}(-n,m_{\ell,p,a,r})$ as components of vector-valued Kloosterman sums. Note that when $u=p$, we have $c_{(p)}=\frac{c}{p}$, $a_{(p)}=a$ and $p_{(a)}=p$ for $p|c$ and $p\nmid a$, hence the formulas from (1.37) to (1.40) become simpler.

Let $\mu_p:\Gamma_0(p)\to \mathrm{GL}_{p-1}(\mathbb{C})$ be defined as in (2.11), $\mathbf{S}_{\infty\infty}(m,n,c,\mu_p)$ and $\mathbf{S}_{0\infty}(\mathbf{X}_r,n,a,\mu_p;r)$ (for $r\geq 0$) be the vector-valued Kloosterman sum defined in (2.43) and (2.48), with $S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)$ and $S^{(\ell)}_{0\infty}(X_r^{([a\ell])},n,a,\mu_p;r)$ as the scalar values at their $\ell$-th entry, respectively. Let $x_r$ be the only root in $(0,\frac12)$ of the quadratic equation

$$\frac{3}{2}x^2-\left(\frac12+r\right)x+\frac{1}{24}=0. \tag{1.42}$$

In the case of prime $p$, we define $[a\ell]$ by $0\leq [a\ell]<p$ and $[a\ell]\equiv a\ell \pmod p$. Then we have the following theorem.

**Theorem 1.14** ([30, Theorem 1.1])**.** *For every prime $p\geq 5$, integer $1\leq \ell\leq p-1$ and positive integer $n$, with the Kloosterman sums defined in (2.44) and (2.49), we have*

$$\begin{aligned}A\left(\frac{\ell}{p};n\right)=&\frac{2\pi e(-\frac18)\sin(\frac{\pi\ell}{p})}{(24n-1)^{\frac14}}\sum_{c>0:\,p|c}\frac{S^{(\ell)}_{\infty\infty}(0,n,c,\mu_p)}{c}I_{\frac12}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right)\\&+\frac{4\pi\sin(\frac{\pi\ell}{p})}{(n-\frac{1}{24})^{\frac14}}\sum_{\substack{r\geq 0\\ x_r^{-1}<p}}\sum_{\substack{a>0:\,p\nmid a,\\ \frac{[a\ell]}{p}\in(0,x_r)\\ \text{or }\frac{[a\ell]}{p}\in(1-x_r,1)}}\frac{S^{(\ell)}_{0\infty}\left(\lceil -p\delta_{\ell,p,a,r}\rceil,n,a,\mu_p;r\right)}{a\cdot\delta_{\ell,p,a,r}^{-\frac14}}I_{\frac12}\left(\frac{4\pi\sqrt{\delta_{\ell,p,a,r}(n-\frac{1}{24})}}{a}\right),\end{aligned} \tag{1.43}$$

*where $\lceil x\rceil$ is the smallest integer $\geq x$ and $\lfloor x\rfloor$ is the largest integer $\leq x$.*

*Remark.* This theorem also proves that Bringmann's formula (1.41), when summing up $c$ and $a$ to infinity, is the exact formula. Indeed, for all prime $p\geq 5$, $1\leq\ell\leq p-1$, $r\geq 0$, positive integers $a,c$ such that $p|c$ and $p\nmid a$, and when $\delta_{\ell,p,a,r}>0$, we have the following relations:

$$\overline{i\cdot B_{\ell,p,c}(-n,0)}=e(-\tfrac18)\sin(\tfrac{\pi\ell}{p})S^{(\ell)}_{\infty\infty}(0,n,c,\mu_p), \tag{1.44}$$

$$\overline{D_{\ell,p,a,r}(-n,m_{\ell,p,a,r})}=S^{(\ell)}_{0\infty}\left(\lceil -p\delta_{\ell,p,a,r}\rceil,n,a,\mu_p;r\right), \tag{1.45}$$

$$I_{\frac12}(z)=(\tfrac{2}{\pi z})^{\frac12}\sinh(z)\quad\text{(see [31, (10.39.1)]), and} \tag{1.46}$$

$$\delta_{\ell,p,a,r}>0\quad\text{if and only if}\quad \tfrac{[a\ell]}{p}\in(0,x_r)\cup(1-x_r,1). \tag{1.47}$$

The last relation (1.47) is clear from the definition. Since $A(\frac{\ell}{p};n)$, $\delta_{\ell,p,a,r}$ and $I_{\frac12}(y)$ (for $y\in\mathbb{R}$) are all real (see (1.32) for $A$), we are safe to take the complex conjugation of (1.41).

We will prove (1.44) and (1.45) in §7.2.

By proving specific vanishing properties of the Kloosterman sums above, we are able to prove the equidistribution of $A(\frac{\ell}{p};n)$ related to the rank of partitions. These properties also provide a new proof of Ramanujan's congruences (1.28) for the $5n+4$ and $7n+5$ cases as described by Dyson.

**Theorem 1.15** ([30, Theorem 1.2])**.** *For all integers $n\geq 0$ and $1\leq\ell\leq p-1$ for $p=5,7$ (mentioned by $p|c$ below), we have the following vanishing conditions for the Kloosterman sums appearing in Theorem 1.14:*

*(1) If* $5|c$*, we have* $S^{(\ell)}_{\infty\infty}(0,5n+4,c,\mu_5)=0$.

*(2) If* $7|c$, $\frac{c}{7}\cdot\ell\not\equiv 1 \pmod 7$*, and* $\frac{c}{7}\cdot\ell\not\equiv -1 \pmod 7$*, then* $S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7;0)=0$.

*(3) If* $7|c$, $7\nmid a$, $a\ell\equiv\pm1 \pmod 7$*, and* $c=7a$*, we have* $\delta_{\ell,7,a,0}=(7^2\times 24)^{-1}$, $\lceil -7\delta_{\ell,7,a,0}\rceil=0$*, and*

$$e(-\tfrac{1}{8})\frac{S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7)}{7}+\frac{2}{\sqrt{7}}S^{(\ell)}_{0\infty}(0,7n+5,a,\mu_7;0)=0.$$

*Remark.* The second sum for $a$ on $r\geq 0$ in (1.41) and (1.43) only appears when $p\geq 7$. When $p=7$, only $r=0$ is allowed and $x_0=\frac{1}{6}$, so $\frac{[a\ell]}{7}$ can only take values $\frac{1}{7}$ or $\frac{6}{7}$, which requires $a\ell\equiv\pm1 \pmod 7$.

**Corollary 1.16** ([30, Corollary 1.3]). *For* $n\geq 0$ *and all* $\ell$, $A(\frac{\ell}{5};5n+4)=A(\frac{\ell}{7};7n+5)=0$.

Combined with (1.32), the above corollary proves the equidistribution properties, i.e. Dyson's conjectures of the rank of partitions: $5N(\ell,5;5n+4)=p(5n+4)$ and $7N(\ell,7;7n+5)=p(7n+5)$ for all $\ell$, which imply Ramanujan's congruences $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$.

## 1.5 Structure of the thesis

The main subject of this thesis is to organize the author's proofs of Theorem 1.7, Theorem 1.9, Theorem 1.14 and Theorem 1.15 as his contribution during the doctoral years. We will introduce the basic definitions of various automorphic forms in Chapter 2. Before we go into the ideas of Theorem 1.7, we first introduce Goldfeld and Sarnak's important work [16] in Chapter 3, including generalizations to vector-valued Kloosterman sums as a help to the proof of Theorem 1.14.

Chapter 4 and Chapter 5 record the author's proof of Theorem 1.7. Although the statement of Theorem 1.7 is the same in the cases $\tilde{m}\tilde{n}<0$ or $\tilde{m}\tilde{n}>0$, the methods we use are quite different, as can be seen by comparing Theorem 4.8 and Theorem 5.1. Since Theorem 1.9 is a corollary of Theorem 1.7, Chapter 6 contains the proof of Theorem 1.9, as well as a literature review section in order to describe the milestones on this route.

Chapter 7 is devoted to prove Theorem 1.14. To prove the vanishing properties of certain Kloosterman sums in Theorem 1.15 and hence the equidistribution properties of $A(\frac{\ell}{p};n)$, Chapter 8 includes many tables to enumerate all the conditions.

# Chapter 2

# Automorphic forms and Kloosterman sums

This chapter includes definitions and basic theorems in the theory of holomorphic modular forms, Maass forms and harmonic Maass forms. We focus on the half-integral weight $k \in \mathbb{Z} + \frac{1}{2}$ unless specified.

## 2.1 Holomorphic modular forms

For any $\gamma \in \mathrm{SL}_2(\mathbb{R})$ and $z = x + iy \in \mathbb{H}$, we have the basic properties

$$\operatorname{Im} \gamma z = \frac{\operatorname{Im} z}{|cz + d|^2} \quad \text{and} \quad \gamma z = \frac{a}{c} - \frac{1}{c(cz + d)}. \tag{2.1}$$

Fixing the argument in $(-\pi, \pi]$, we now define the holomorphic modular forms.

**Definition 2.1.** *Let $k \in \mathbb{Z} + \frac{1}{2}$ and $\nu$ be a weight $k$ multiplier system on the congruence subgroup $\Gamma$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a holomorphic modular form of weight $k$ for $(\Gamma, \nu)$ if it satisfies:*

*(1) $f(\gamma z) = \nu(\gamma)(cz + d)^k f(z)$ for all $\gamma \in \Gamma$;*

*(2) $f$ is holomorphic at the cusps of $\Gamma$.*

*Moreover, if $f$ also satisfies*

*(3) $f$ vanishes at all the cusps of $\Gamma$,*

*then $f$ is called a holomorphic cusp form of weight $k$ for $(\Gamma, \nu)$.*

We denote $M_k(\Gamma, \nu)$ (resp. $S_k(\Gamma, \nu)$) as the space of holomorphic modular (resp. cusp) form of weight $k$ for $(\Gamma, \nu)$. Recall $\tilde{n} = n - \alpha_\nu$. For $f \in M_k(\Gamma, \nu)$, $f$ has a Fourier expansion at the cusp $\infty$ given by

$$f(z) = \sum_{n=0}^{\infty} a_f(n) e(\tilde{n} z). \tag{2.2}$$

We call $a_f(n)$ the Fourier coefficient of $f$ (at the cusp $\infty$). For any cusp $\mathfrak{a}$ of $\Gamma$, let $\sigma_{\mathfrak{a}} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be the scaling matrix (1.10) of $\mathfrak{a}$. Then we can write the Fourier expansion of $f$ at the cusp $\mathfrak{a}$ as

$$(Cz + D)^{-k} f(\sigma_{\mathfrak{a}} z) = \sum_{n=0}^{\infty} a_{f,\mathfrak{a}}(n) e(n_{\mathfrak{a}} z). \tag{2.3}$$

Therefore, $f \in S_k(\Gamma, \nu)$ if and only if $a_{f,\mathfrak{a}}(0) = 0$ for all cusps $\mathfrak{a}$ of $\Gamma$. The spaces $M_k(\Gamma, \nu)$ and $S_k(\Gamma, \nu)$ are both finite dimensional.

The hyperbolic measure $d\mu(z)$ on $\mathbb{H}$ is defined by

$$d\mu(z) = \frac{dxdy}{y^2} \tag{2.4}$$

where $dx$ and $dy$ are the usual Lebesgue measures. One can check that for all $\gamma \in \mathrm{GL}_2(\mathbb{R})$, we always have $d\mu(\gamma z) = d\mu(z)$, i.e. $d\mu(z)$ is invariant under the action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathbb{H}$. Therefore, for $f, g \in M_k(\Gamma, \nu)$, the following measure

$$y^k f(z)\overline{g(z)} d\mu(z)$$

is invariant under the action of $\Gamma$ on $\mathbb{H}$ (briefly called $\Gamma$-invariant). We define the Petersson inner product for holomorphic forms as

$$\langle f, g \rangle := \int_{\Gamma \backslash \mathbb{H}} y^k f(z)\overline{g(z)} \frac{dxdy}{y^2}. \tag{2.5}$$

When either $f$ or $g$ is in $S_k(\Gamma, \nu)$, the integral converges absolutely. The linear space $S_k(\Gamma, \nu)$ is then a finitely dimensional Hilbert space with the Petersson inner product.

### 2.1.1 Holomorphic cusp forms of half-integral weight

Recall that we have already defined the two weight $\frac{1}{2}$ multiplier systems: $\nu_\theta$ (1.15) and $\nu_\eta$ (1.18). Let $r(\psi)$ denote the conductor of a Dirichlet character $\psi$. When the weight $k = \frac{1}{2}$, we have the Serre-Stark basis theorem:

**Theorem 2.2** ([32, Corollary 1 of Theorem A]). *The space $M_{\frac{1}{2}}(\Gamma_1(N), \nu_\theta)$ has a basis consisting of*

$$\theta_{\psi,t}(z) := \sum_{n \in \mathbb{Z}} \psi(n) q^{tn^2}$$

*where $\psi$ is an even primitive Dirichlet character whose conductor $r(\psi)$ satisfies $4r(\psi)^2 t | N$.*

For positive integers $N, l$ and a weight $k \in \mathbb{Z} + \frac{1}{2}$ multiplier $\nu$ on $\Gamma_0(N)$, we know that $\nu$ is also a weight $k + 2l$ multiplier system on $\Gamma_0(N)$. For simplicity we denote $K = k + 2l \in \mathbb{Z} + \frac{1}{2}$. Recall that $S_K(\Gamma_0(N), \nu)$ is a finite-dimensional Hilbert space under the Petersson inner product. If we take an orthonormal basis $\{F_j(\cdot) :\ 1 \le j \le d := \dim S_K(\Gamma_0(N), \nu)\}$ of $S_K(\Gamma_0(N), \nu)$ and write the Fourier expansion of $F_j$ as

$$F_j(z) = \sum_{n=1}^{\infty} a_j(n) e(\tilde{n} z),$$

then we have the Petersson trace formula

$$\frac{\Gamma(K-1)}{(4\pi\tilde{n})^{K-1}}\sum_{j=1}^{d}|a_j(n)|^2 = 1 + 2\pi i^{-K}\sum_{N|c}\frac{S(n,n,c,\nu)}{c}J_{K-1}\left(\frac{4\pi\tilde{n}}{c}\right). \tag{2.6}$$

The left hand side is independent from the choice of the basis.

## 2.2 Maass forms

In this section we recall some basic facts about Maass forms with general weight and multiplier, which can be found in various references like [11], [12], [33]–[35]. Let $\Gamma$ denote our congruence subgroup with $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in\Gamma$ and $\mathbb{H}$ denote the upper-half complex plane. Recall the definition of $j(\gamma,z)$ in (1.7) and the definition of the weight $k\in\mathbb{Z}+\frac{1}{2}$ slash operator defined in (1.8). We call a function $f:\mathbb{H}\to\mathbb{C}$ automorphic of weight $k$ and multiplier $\nu$ on $\Gamma$ if

$$f|_k\gamma = \nu(\gamma)f \quad \text{for all } \gamma\in\Gamma.$$

Let $\mathcal{A}_k(\Gamma,\nu)$ denote the linear space consisting of all such functions and $\mathcal{L}_k(\Gamma,\nu)\subset\mathcal{A}_k(\Gamma,\nu)$ denote the space of square-integrable functions on $\Gamma\setminus\mathbb{H}$ with respect to the measure

$$d\mu(z) = \frac{dxdy}{y^2}$$

and the Petersson inner product

$$\langle f,g\rangle := \int_{\Gamma\setminus\mathbb{H}} f(z)\overline{g(z)}\frac{dxdy}{y^2}$$

for $f,g\in\mathcal{L}_k(\Gamma,\nu)$. For $k\in\mathbb{R}$, the Laplacian

$$\Delta_k := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x} \tag{2.7}$$

can be expressed as

$$\Delta_k = -R_{k-2}L_k - \frac{k}{2}\left(1-\frac{k}{2}\right) \tag{2.8}$$

$$= -L_{k+2}R_k + \frac{k}{2}\left(1+\frac{k}{2}\right) \tag{2.9}$$

where $R_k$ is the Maass raising operator

$$R_k := \frac{k}{2} + 2iy\frac{\partial}{\partial z} = \frac{k}{2} + iy\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \tag{2.10}$$

and $L_k$ is the Maass lowering operator

$$L_k := \frac{k}{2} + 2iy\frac{\partial}{\partial \bar{z}} = \frac{k}{2} + iy\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{2.11}$$

These operators raise and lower the weight of an automorphic form as

$$(R_kf)|_{k+2}\,\gamma = R_k(f|_k\gamma), \quad (L_kf)|_{k-2}\,\gamma = L_k(f|_k\gamma), \quad \text{for } f\in\mathcal{A}_k(\Gamma,\nu)$$

and satisfy the commutative relations

$$R_k\Delta_k = \Delta_{k+2}R_k, \quad L_k\Delta_k = \Delta_{k-2}L_k. \tag{2.12}$$

Moreover, $\Delta_k$ commutes with the weight $k$ slash operator for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

We call a real analytic function $f : \mathbb{H} \to \mathbb{C}$ an eigenfunction of $\Delta_k$ with eigenvalue $\lambda \in \mathbb{C}$ if

$$\Delta_k f + \lambda f = 0.$$

From (2.12), it is clear that an eigenvalue $\lambda$ for the weight $k$ Laplacian is also an eigenvalue for weight $k \pm 2$. We call an eigenfunction $f$ a Maass form if $f \in \mathcal{A}_k(\Gamma,\nu)$ is smooth and satisfies the growth condition

$$(f|_k\gamma)(x+iy) \ll y^\sigma + y^{1-\sigma}$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and some $\sigma$ depending on $\gamma$ when $y \to +\infty$. Moreover, if a Maass form $f$ satisfies

$$\int_0^1 (f|_k\sigma_{\mathfrak{a}})(x+iy)\, e(\alpha_{\nu,\mathfrak{a}}x)dx = 0$$

for all cusps $\mathfrak{a}$ of $\Gamma$, then $f \in \mathcal{L}_k(\Gamma,\nu)$ and we call $f$ a Maass cusp form. For details see [11, §2.3]

Let $\mathcal{B}_k(\Gamma,\nu) \subset \mathcal{L}_k(\Gamma,\nu)$ denote the space of smooth functions $f$ such that both $f$ and $\Delta_k f$ are bounded. One can show that $\mathcal{B}_k(\Gamma,\nu)$ is dense in $\mathcal{L}_k(\Gamma,\nu)$ and $\Delta_k$ is self-adjoint on $\mathcal{B}_k(\Gamma,\nu)$. If we let $\lambda_0 := \lambda_0(k) = \frac{|k|}{2}(1-\frac{|k|}{2})$, then for $f \in \mathcal{B}_k(\Gamma,\nu)$,

$$\langle f, -\Delta_k f\rangle \ge \lambda_0\langle f, f\rangle,$$

i.e. $-\Delta_k$ is bounded from below. By the Friedrichs extension theorem, $-\Delta_k$ can be extended to a self-adjoint operator on $\mathcal{L}_k(\Gamma,\nu)$. The spectrum of $\Delta_k$ consists of two parts: the continuous spectrum $\lambda \in [\frac{1}{4},\infty)$ and a discrete spectrum of finite multiplicity contained in $[\lambda_0,\infty)$.

Non-zero eigenfunctions corresponding to eigenvalue $\lambda_0$ come from holomorphic modular forms. To be precise, let $M_k(\Gamma,\nu)$ denote the space of holomorphic modular forms of weight $k$ and multiplier $\nu$ on $\Gamma$. There is a one-to-one correspondence between all $f \in \mathcal{L}_k(\Gamma,\nu)$ with eigenvalue $\lambda_0$ and weight $k$ holomorphic modular forms $F$ by

$$f(z) = \begin{cases} y^{\frac{k}{2}}F(z) & k \ge 0,\ F \in M_k(\Gamma,\nu), \\ y^{-\frac{k}{2}}\overline{F(z)} & k < 0,\ F \in M_{-k}(\Gamma,\overline{\nu}). \end{cases} \tag{2.13}$$

For the Fourier expansion $\sum_{n\in\mathbb{Z}} a_y(n)e(\tilde{n}x)$ of such $f$, we have

$$\begin{cases} k \ge 0 & \Rightarrow \quad a_y(n) = 0 \text{ for } \tilde{n} < 0, \\ k < 0 & \Rightarrow \quad a_y(n) = 0 \text{ for } \tilde{n} > 0. \end{cases} \tag{2.14}$$

Let $\lambda_\Delta(\Gamma,\nu,k)$ denote the first eigenvalue larger than $\lambda_0$ in the discrete spectrum with respect to $\Gamma$, weight $k$ and multiplier $\nu$. For weight 0, Selberg showed that $\lambda_\Delta(\Gamma(N),\mathbf{1},0) \ge \frac{3}{16}$ for all $N$ [23] and Selberg's famous eigenvalue conjecture states that $\lambda_\Delta(\Gamma,\mathbf{1},0) \ge \frac{1}{4}$ for all $\Gamma$. We introduce the hypothesis $H_\theta$ as

$$H_\theta: \quad \lambda_\Delta(\Gamma_0(N),\mathbf{1},0) \ge \tfrac{1}{4} - \theta^2 \quad \text{for all } N. \tag{2.15}$$

Selberg's conjecture includes $H_0$ and the best progress known today is $H_{\frac{7}{64}}$ by [25]. We denote $\lambda_\Delta(G,\nu,k)$ as

$\lambda_\Delta$ when $(G, \nu, k)$ is clear from context.

Let $\tilde{\mathcal{L}}_k(\Gamma, \nu) \subset \mathcal{L}_k(\Gamma, \nu)$ denote the subspace spanned by eigenfunctions of $\Delta_k$. For each eigenvalue $\lambda$, we write

$$\lambda = \tfrac{1}{4} + r^2 = s(1-s), \quad s = \tfrac{1}{2} + ir, \quad r \in i(0, \tfrac{1}{4}] \cup [0, \infty).$$

So $r \in i\mathbb{R}$ corresponds to $\lambda < \frac{1}{4}$ and any such $\lambda \in (\lambda_0, \frac{1}{4})$ is called an exceptional eigenvalue. Set

$$r_\Delta(N, \nu, k) := i \cdot \sqrt{\tfrac{1}{4} - \lambda_\Delta(\Gamma_0(N), \nu, k)}. \tag{2.16}$$

Let $\tilde{\mathcal{L}}_k(\Gamma, \nu, r) \subset \tilde{\mathcal{L}}_k(\Gamma, \nu)$ denote the subspace corresponding to the spectral parameter $r$. Complex conjugation gives an isometry

$$\tilde{\mathcal{L}}_k(\Gamma, \nu, r) \longleftrightarrow \tilde{\mathcal{L}}_{-k}(\Gamma, \overline{\nu}, r)$$

between normed spaces. For each $v \in \tilde{\mathcal{L}}_k(n, \nu, r)$, we have the Fourier expansion

$$v(z) = v(x+iy) = c_0(y) + \sum_{\tilde{n} \neq 0} \rho(n) W_{\frac{k}{2} \operatorname{sgn} \tilde{n},\, ir}(4\pi|\tilde{n}|y) e(\tilde{n}x)$$

where $W_{\kappa,\mu}$ is the Whittaker function as in [31, (13.14.3)] and

$$c_0(y) = \begin{cases} 0 & \alpha_\nu \neq 0, \\ 0 & \alpha_\nu = 0 \text{ and } r \geq 0, \\ \rho(0) y^{\frac{1}{2}+ir} & \alpha_\nu = 0 \text{ and } r \in i(0, \frac{1}{4}]. \end{cases}$$

Using the fact that $W_{\kappa,\mu}$ is a real function when $\kappa$ is real and $\mu \in \mathbb{R} \cup i\mathbb{R}$ [31, (13.4.4), (13.14.3), (13.14.31)], if we denote the Fourier coefficient of $f_c := \bar{f}$ as $\rho_c(n)$, then

$$\rho_c(n) = \begin{cases} \overline{\rho(1-n)}, & \alpha_\nu > 0,\ n \neq 0 \\ \overline{\rho(-n)}, & \alpha_\nu = 0. \end{cases} \tag{2.17}$$

## 2.3 Harmonic Maass forms

The following construction can be found in [7], [21]. Let $k \in \frac{1}{2} + \mathbb{Z}$, $z = x + iy$ for $x, y \in \mathbb{R}$ and $y \neq 0$, $s \in \mathbb{C}$, $4|N$ is a positive integer. We define the weight $k$ hyperbolic Laplacian (different from the former section) by

$$\widetilde{\Delta}_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

**Definition 2.3.** *With the notations above, let $\chi$ be a Dirichlet character modulo $N$. A weight $k$ harmonic Maass form on $\Gamma_0(N)$ with Nebentypus $\chi$ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:*

*(1) For all $\gamma \in \Gamma_0(N)$, we have $f(\gamma z) = \chi(d) \nu_\theta(\gamma)^{2k} (cz+d)^k f(z)$;*

*(2) $\widetilde{\Delta}_k f = 0$;*

*(3) There exists a polynomial $\mathcal{P}(z) = \sum_{n \leq 0} a^+(n) q^n$ with coefficients in $\mathbb{C}$ such that*

$$f(z) - \mathcal{P}(z) = O(e^{-Cy})$$

*for some $C > 0$. Analogous conditions are required for all cusps.*

*Remark.* We denote the space of such harmonic Maass forms by $H_k(N, \chi\nu_\theta^{2k})$. The polynomial $\mathcal{P}$ is called the principal part of $f$ at the cusp $\infty$, with analogous definition at other cusps. When the transformation formula in condition (1) is replaced by $f(\gamma z) = \nu(\gamma)(cz+d)^k f(z)$ for some multiplier system $\nu$, we call $f$ a weight $k$ harmonic Maass form for $(\Gamma_0(N), \nu)$ and the principal parts of $f$ are defined similarly for cusps of $\Gamma_0(N)$.

Denote the anti-linear differential operator $\xi_k$ by

$$(\xi_k g)(z) := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}(g(z))} = R_{-k}(y^k \overline{g(z)})$$

where $R_k$ is the Maass raising operator defined in (2.12). If we let $G(z) = g(Bz)$ for some constant B, one can check that $(\xi_k G)(z) = B^{1-k}(\xi_k g)(Bz)$. The following lemma is crucial in Chapters 6 and 7:

**Lemma 2.4** ([36, Proposition 3.2],[21, Lemma 2.2])**.** *The map*

$$\xi_k : H_k(N, \chi\nu_\theta^{2k}) \to S_{2-k}(N, \overline{\chi}\nu_\theta^{-2k})$$

*is a surjective map. Moreover, if $f \in H_k(N, \chi)$ has Fourier expansion*

$$f(z) = \sum_{n \geq n_0} c_f^+(n) q^n + \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n \quad \text{for some } n_0 \in \mathbb{Z},$$

*then*

$$(\xi_k f)(z) = -(4\pi)^{1-k} \sum_{n=1}^{\infty} \overline{c_f^-(-n)} n^{1-k} q^n.$$

*Remark.* We denote the holomorphic part of $f$ by

$$f_h(z) := \sum_{n \geq n_0} c_f^+(n) q^n = \mathcal{P}(z) + \sum_{n>0} c_f^+(n) q^n$$

and the non-holomorphic part of $f$ by

$$f_{nh}(z) := \sum_{n<0} c_f^-(n) \Gamma(1-k, 4\pi|n|y) q^n,$$

where $\Gamma(s, \beta)$ is the incomplete Gamma function defined by

$$\Gamma(s, \beta) = \int_\beta^\infty t^{s-1} e^{-t} dt, \quad \beta > 0.$$

We also define a mock modular form as the holomorphic part of a harmonic Maass form.

We define the following functions to prepare our constructions of harmonic Maass forms later. Denote $M_{\beta,\mu}$ and $W_{\beta,\mu}$ as the $M$- and $W$-Whittaker functions defined at [31, (13.14.2-3)]. For $s \in \mathbb{C}$, $x, y \in \mathbb{R}$, and $k \in \mathbb{Z} + \frac{1}{2}$, we define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn} y,\, s-\frac{1}{2}}(|y|) \quad \text{and} \quad \varphi_{s,k}(x+iy) := \mathcal{M}_s(4\pi y) e(x). \tag{2.18}$$

We also define

$$\mathcal{W}_s(y) := |y|^{-\frac{k}{2}} W_{\frac{k}{2}\operatorname{sgn} y,\, s-\frac{1}{2}}(|y|). \tag{2.19}$$

These functions have the following properties. For $y > 0$, by [31, (13.18.4)] we have

$$\mathcal{M}_{1-\frac{k}{2}}(-y) = y^{-\frac{k}{2}} M_{-\frac{k}{2},\,\frac{1}{2}-\frac{k}{2}}(y) = (1-k)\left(\Gamma(1-k) - \Gamma(1-k,y)\right) e^{\frac{y}{2}}, \tag{2.20}$$

and by [31, (13.18.2)], we have

$$W_{-\frac{k}{2},\,\frac{1}{2}-\frac{k}{2}}(y) = y^{\frac{k}{2}} e^{\frac{y}{2}} \Gamma(1-k,y) \quad \text{and} \quad W_{\frac{k}{2},\,\frac{1}{2}-\frac{k}{2}}(y) = y^{\frac{k}{2}} e^{-\frac{y}{2}}. \tag{2.21}$$

Moreover, $\varphi_{s,k}(z)$ is an eigenfunction of $\widetilde{\Delta}_k$ with eigenvalue $s(1-s) + \frac{k^2-2k}{4}$. Specifically, when $s = 1 - \frac{k}{2}$, we have

$$\widetilde{\Delta}_k \varphi_{1-\frac{k}{2},k} = 0. \tag{2.22}$$

## 2.4 Vector-valued theory

### 2.4.1 Vector-valued Maass forms

Analogous to the scalar-valued case and [37] for vector-valued modular forms, here we define vector-valued Maass forms on a congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ where $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.

For a vector or a matrix $M$, let $M^{\mathrm{T}}$ denote its transpose and $M^{\mathrm{H}}$ denote its conjugate transpose (Hermitian). We clarify the notations for the remaining part of this thesis here. Note that we are not using the language of Weil representations.

**Notation 2.5.** *The boldface letter, e.g.* $\mathbf{u}$ *or* $\mathbf{F}(z)$*, always denotes a vector or a vector-valued function of some dimension* $D \geq 2$*, respectively. For* $1 \leq \ell \leq D$*, let* $\mathfrak{e}_\ell := (0, \cdots, 0, 1, 0, \cdots, 0)^{\mathrm{T}}$ *denote the unit vector which has* $1$ *at its* $\ell$*-th entry and* $0$ *at the others.*

*When the superscript* $\cdot^{(\ell)}$ *appears,* $\mathbf{u}^{(\ell)}$*,* $u^{(\ell)}$*,* $\mathbf{F}^{(\ell)}(z)$ *and* $F^{(\ell)}(z)$ *are defined by*

$$\mathbf{u} = \sum_{\ell=1}^{D} \mathbf{u}^{(\ell)} = (u^{(1)}, u^{(2)}, \cdots, u^{(D)})^{\mathrm{T}}, \quad \mathbf{F}(z) = \sum_{\ell=1}^{D} \mathbf{F}^{(\ell)}(z) = \sum_{\ell=1}^{D} F^{(\ell)}(z)\mathfrak{e}_\ell,$$

*where* $\mathbf{u}^{(\ell)} = u^{(\ell)}\mathfrak{e}_\ell$ *and* $\mathbf{F}^{(\ell)}(z) = F^{(\ell)}(z)\mathfrak{e}_\ell$*.*

Given two $D$-dimensional complex vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^D$, we define their inner product as $\mathbf{v}^{\mathrm{H}}\mathbf{u} = \sum_{\ell=1}^{D} u^{(\ell)}\overline{v^{(\ell)}}$. Let $\mathrm{M}_D(\mathbb{C})$ denote the space of $D \times D$ complex matrices and take $M \in \mathrm{M}_D(\mathbb{C})$. Then the inner product of $M\mathbf{v}$ and $\mathbf{u}$ is

$$(M\mathbf{v})^{\mathrm{H}}\mathbf{u} = \mathbf{v}^{\mathrm{H}} M^{\mathrm{H}} \mathbf{u}.$$

Recall that in the scalar-valued case, we fix the argument $(-\pi, \pi]$ and define the automorphic factor $j(\gamma, z)$ in (1.7). Denote our vector-valued function on the upper-half complex plane $\mathbb{H}$ by

$$\mathbf{F}(z) = (F^{(1)}(z), F^{(2)}(z), \cdots, F^{(D)}(z))^T = \sum_{\ell=1}^{D} F^{(\ell)}(z)\mathfrak{e}_\ell$$

and define the weight $k \in \mathbb{R}$ slash operator $|_k$ by

$$(\mathbf{F}|_k\gamma)(z) := \Big((F^{(1)}|_k\gamma)(z), \cdots, (F^{(D)}|_k\gamma)(z)\Big)^T := j(\gamma, z)^{-k}\mathbf{F}(\gamma z).$$

**Definition 2.6.** *For a congruence subgroup $\Gamma$ of $\mathrm{SL}_2(\mathbb{Z})$ with $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\in\Gamma$, we say that $\xi:\Gamma\to\mathrm{GL}_D(\mathbb{C})$ is a $D$-dimensional multiplier system if it satisfies the following compatibility conditions:*

*(1) $\xi(\gamma)$ is a unitary matrix for all $\gamma\in\Gamma$, i.e. $\xi(\gamma)^{-1}=\xi(\gamma)^{\mathrm{H}}$.*

*(2) $\xi(-I)=e^{-\pi ik}I_D$. Here $I$ is the identity matrix in $\mathrm{SL}_2(\mathbb{Z})$ and $I_D$ is the identity matrix in $\mathrm{GL}_D(\mathbb{C})$.*

*(3) $\xi(\gamma_1\gamma_2)=w_k(\gamma_1,\gamma_2)\xi(\gamma_1)\xi(\gamma_2)$ for all $\gamma_1,\gamma_2\in\Gamma$, where*

$$w_k(\gamma_1,\gamma_2):=j(\gamma_2,z)^kj(\gamma_1,\gamma_2z)^kj(\gamma_1\gamma_2,z)^{-k}.$$

*(4) For every cusp $\mathfrak{a}$ of $\Gamma$, $\xi\left(\sigma_{\mathfrak{a}}\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_{\mathfrak{a}}^{-1}\right)=\mathrm{diag}\{e(-\alpha_{\xi,\mathfrak{a}}^{(1)}),\cdots,e(-\alpha_{\xi,\mathfrak{a}}^{(D)})\}$ for some $\alpha_{\xi,\mathfrak{a}}^{(\ell)}\in[0,1)$. Here $\sigma_{\mathfrak{a}}$ is the scaling matrix of the cusp $\mathfrak{a}$ of $\Gamma$.*

*Remark.* The multiplier system $\xi$ may not be a matrix representation of $\Gamma$ because when $k\notin\mathbb{Z}$, $w_k(\gamma_1,\gamma_2)$ may not always be 1, hence $\xi$ is not multiplicative. In (4), if $\xi$ is clear from context, we will simply denote $\alpha_{\mathfrak{a}}^{(\ell)}=\alpha_{\xi,\mathfrak{a}}^{(\ell)}$.

For the weight $k\in\mathbb{R}$ and a $D$-dimensional complex function $\mathbf{F}$ such that each component $F^{(\ell)}$ is a smooth function, if

$$\Delta_k\mathbf{F}(z)+\lambda\mathbf{F}(z)=0$$

for some $\lambda\in\mathbb{C}$, then we call $\mathbf{F}$ a $D$-dimensional eigenfunction of $\Delta_k$ with eigenvalue $\lambda$. In this case, every component $F^{(\ell)}$ of $\mathbf{F}$ is a eigenfunction of $\Delta_k$ with eigenvalue $\lambda$.

**Definition 2.7.** *A vector-valued Maass form $\mathbf{F}:\mathbb{H}\to\mathbb{C}^D$ of weight $k\in\mathbb{R}$, eigenvalue $\lambda\in\mathbb{C}$ and $D$-dimensional multiplier system $\xi$ on $\Gamma$ is a vector-valued function $\mathbf{F}=(F^{(1)},\cdots,F^{(D)})$ satisfying:*

*(1) Each $F^{(\ell)}$ is real-analytic on $\mathbb{H}$;*

*(2) $(\mathbf{F}|_k\gamma)(z)=\xi(\gamma)\mathbf{F}(z)$ for all $\gamma\in\Gamma$.*

*(3) $(\Delta_k+\lambda)\mathbf{F}(z)=0$.*

*(4) Each $F^{(\ell)}$ satisfies the growth condition*

$$(F^{(\ell)}|_k\gamma)(z)\ll y^\sigma+y^{1-\sigma}\quad\text{for some }\sigma>0,\quad\text{as }y\to\infty$$

*for all $\gamma\in\mathrm{SL}_2(\mathbb{Z})$.*

*In addition, if $\mathbf{F}$ also satisfies*

*(5) For every $1\le\ell\le D$ and every cusp $\mathfrak{a}$ of $\Gamma$*

$$\int_0^1(F^{(\ell)}|_k\sigma_{\mathfrak{a}})(x+iy)\,e(\alpha_{\mathfrak{a}}^{(\ell)}x)dx=0,$$

*then we call $\mathbf{F}$ a $D$-dimensional Maass cusp form.*

Suppose $\mathbf{F}$ is a weight $k$ Maass form with multiplier system $\xi$ on $\Gamma$. By Definition 2.7, each $e(\alpha_{\mathfrak{a}}^{(\ell)})(F^{(\ell)}|_k\sigma_{\mathfrak{a}})(z)$ is a periodic function with period 1 on $\mathbb{H}$. Then $F^{(\ell)}$ admits a Fourier expansion at the cusp $\mathfrak{a}$ as

$$(F^{(\ell)}|_k\sigma_{\mathfrak{a}})(x+iy) = \sum_{n\in\mathbb{Z}} c_{\mathfrak{a}}^{(\ell)}(n,y)e\left(n_{\mathfrak{a}}^{(\ell)}x\right), \quad \text{where } n_{\mathfrak{a}}^{(\ell)} := n - \alpha_{\mathfrak{a}}^{(\ell)}.$$

As in the classical case, since $F^{(\ell)}$ is an eigenfunction of $\Delta_k$ with eigenvalue $\lambda = \frac{1}{4}+r^2$ for $r \in [0,\infty)\cup i[0,\infty)$, by solving the partial differential equation using the method of separation of variables, $F^{(\ell)}$ admits a Fourier expansion of the form

$$F^{(\ell)}(x+iy) = \rho_F^{(\ell)}(0,y) + \sum_{\substack{n\in\mathbb{Z}\\ n_\infty^{(\ell)}\neq 0}} \rho_\infty^{(\ell)}(n) W_{\frac{k}{2}\operatorname{sgn} n_\infty^{(\ell)},\, ir}(4\pi|n_\infty^{(\ell)}|y)e\left(n_\infty^{(\ell)}x\right), \tag{2.23}$$

where $\rho_F^{(\ell)}(0,y) = 0$ if $n \neq 0$ or $\alpha_\infty^{(\ell)} \neq 0$, and $\rho_F^{(\ell)}(0,y) = c_1y^{\frac{1}{2}+ir} + c_2y^{\frac{1}{2}-ir}$ for some constants $c_1, c_2 \in \mathbb{C}$ if $n = \alpha_\infty^{(\ell)} = 0$. Here $W_{\kappa,\mu}$ is the $W$-Whittaker function defined at [31, (13.14.3)] which satisfies $W_{\kappa,\mu}(\alpha) \in \mathbb{R}$ and $W_{\kappa,\mu} = W_{\kappa,-\mu}$ if $\kappa, \alpha \in \mathbb{R}$ and $\mu \in \mathbb{R}\cup i\mathbb{R}$.

We call $\mathbf{F} : \mathbb{H} \to \mathbb{C}^D$ a vector-valued automorphic form of weight $k$ and $D$-dimensional multiplier system $\xi$ on $\Gamma$ if

$$(\mathbf{F}|_k\gamma)(z) = \xi(\gamma)\mathbf{F}(z) \quad \text{for all } \gamma \in \Gamma. \tag{2.24}$$

and denote the linear space of all such automorphic forms as $\mathcal{A}_k(\Gamma,\xi)$. For $\mathbf{F}, \mathbf{G} \in \mathcal{A}_k(\Gamma,\xi)$, we define (formally) their Petersson inner product by

$$\langle \mathbf{F}, \mathbf{G}\rangle := \int_{\Gamma\backslash\mathbb{H}} \sum_{\ell=1}^{D} F^{(\ell)}(z)\overline{G^{(\ell)}(z)}\frac{dxdy}{y^2} = \int_{\Gamma\backslash\mathbb{H}} \mathbf{G}^{\mathrm{H}}(z)\mathbf{F}(z)\frac{dxdy}{y^2}. \tag{2.25}$$

This inner product is well-defined: for all $\gamma \in \Gamma$, since $\frac{dxdy}{y^2}$ is invariant under $\gamma$, we have

$$\int_{\Gamma\backslash\mathbb{H}} \mathbf{G}^{\mathrm{H}}(\gamma z)\mathbf{F}(\gamma z)\frac{dxdy}{y^2} = \int_{\Gamma\backslash\mathbb{H}} \mathbf{G}^{\mathrm{H}}(z)\xi(\gamma)^{\mathrm{H}}\overline{j(\gamma,z)^k}j(\gamma,z)^k\xi(\gamma)\mathbf{F}(z)\frac{dxdy}{y^2} = \langle \mathbf{F}, \mathbf{G}\rangle.$$

Let $\mathcal{L}_k(\Gamma,\xi) \subset \mathcal{A}_k(\Gamma,\xi)$ denote the Hilbert space of square-integrable functions under the above inner product. Then if $\mathbf{F} \in \mathcal{L}_k(\Gamma,\xi)$, we have

$$\int_{\Gamma\backslash\mathbb{H}} |F^{(\ell)}(z)|^2\frac{dxdy}{y^2} < \infty \quad \text{for } 1 \le \ell \le D.$$

### 2.4.2 A representation on $\Gamma_0(p)$ twisted by $\overline{\nu_\eta}$

In this section we review the notations and results in [18]. Denote $q = e^{2\pi iz}$ for $z \in \mathbb{H}$ and the $q$-Pochhammer symbol $(a,q)_n := \prod_{j=0}^{n-1}(1-aq^{j-1})$. Define

$$M\left(\frac{\ell}{p};z\right) := \frac{1}{(q,q)_\infty}\sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n+\frac{\ell}{p}}}{1-q^{n+\frac{\ell}{p}}} q^{\frac{3}{2}n(n+1)} \tag{2.26}$$

and

$$N\left(\frac{\ell}{p};z\right):=\frac{1}{(q,q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(1+q^n)\left(2-2\cos(\frac{2\pi\ell}{p})\right)}{1-2\cos(\frac{2\pi\ell}{p})q^n+q^{2n}}q^{\frac{1}{2}n(3n+1)}\right). \tag{2.27}$$

Further define

$$\mathcal{M}\left(\frac{\ell}{p};z\right):=2q^{\frac{3\ell}{2p}(1-\frac{\ell}{p})-\frac{1}{24}}M\left(\frac{\ell}{p};z\right),\quad \mathcal{N}\left(\frac{\ell}{p};z\right):=\csc\left(\frac{\pi\ell}{p}\right)q^{-\frac{1}{24}}N\left(\frac{\ell}{p};z\right), \tag{2.28}$$

We also define the following functions: $\mathcal{M}(a,b,p,z)$ and $\mathcal{N}(a,b,p,z)$ as [18, (2.7), (2.8)]; non-holomorphic functions $T_1(\frac{\ell}{p};z)$, $T_2(\frac{\ell}{p};z)$, $T_1(a,b,p;z)$ and $T_2(a,b,p;z)$ as [18, (3.1)-(3.4)];

$$\varepsilon_2\left(\frac{\ell}{p};z\right):=\begin{cases}2\exp\left(-3\pi iz(\frac{a}{c}-\frac{1}{6})^2\right), & 0<\frac{\ell}{p}<\frac{1}{6},\\ 0, & \frac{1}{6}<\frac{\ell}{p}<\frac{5}{6},\\ 2\exp\left(-3\pi iz(\frac{a}{c}-\frac{5}{6})^2\right), & \frac{5}{6}<\frac{\ell}{p}<1\end{cases} \tag{2.29}$$

and $\varepsilon_2(a,b,p;z)$ as [18, before Theorem 2.4]; and

$$\mathcal{G}_1\left(\frac{\ell}{p};z\right):=\mathcal{N}\left(\frac{\ell}{p};z\right)-T_1\left(\frac{\ell}{p};z\right), \tag{2.30}$$

$$\mathcal{G}_2\left(\frac{\ell}{p};z\right):=\mathcal{M}\left(\frac{\ell}{p};z\right)+\varepsilon_2\left(\frac{\ell}{p};z\right)-T_2\left(\frac{\ell}{p};z\right), \tag{2.31}$$

$$\mathcal{G}_1(a,b,p;z):=\mathcal{N}(a,b,p;z)-T_1(a,b,p;z), \tag{2.32}$$

$$\mathcal{G}_2(a,b,p;z):=\mathcal{M}(a,b,p;z)+\varepsilon_2(a,b,p;z)-T_2(a,b,p;z) \tag{2.33}$$

as [18, (3.5)-(3.8)]. Bringmann and Ono proved the following result in 2010.

**Proposition 2.8** ([8, Theorem 3.4], [18, Corollary 3.2])**.** *Suppose $p\geq 5$ is a prime. Then*

$$\left\{\mathcal{G}_1\left(\frac{\ell}{p};z\right),\mathcal{G}_2\left(\frac{\ell}{p};z\right):1\leq\ell<p\right\}\cup\left\{\mathcal{G}_1(a,b,p;z),\mathcal{G}_1(a,b,p;z):0\leq a<p,\ 1\leq b<p\right\}$$

*is a vector valued Maass form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$.*

We clarify the notations to use in the remaining part of this thesis:

**Notation 2.9.** *For integers $A$ and $n>0$, let $[A]_{\{n\}}$ denote the least non-negative residue of $A \pmod n$, i.e. $0\leq[A]_{\{n\}}<n$ defined by $[A]_{\{n\}}\equiv A \pmod n$. If the prime $p\geq 5$ is clear from context, then we simply denote $[A]_{\{p\}}$ as $[A]$.*

*If $(A,n)=1$, let $\overline{A_{\{n\}}}$ denote the inverse of $A$ modulo $n$, i.e. defined by $A\overline{A_{\{n\}}}\equiv 1 \pmod n$. Let $A'_n$ be defined by $AA'_n\equiv -1 \pmod n$ if $n$ is odd and $AA'_n\equiv -1 \pmod{2n}$ if $n$ is even.*

Garvan computed the following transformation laws on $\Gamma_0(p)$:

**Proposition 2.10** ([18, Theorem 4.1])**.** *Let $p\geq 5$ be a prime. Then*

$$\mathcal{G}_1\left(\frac{\ell}{p};\gamma z\right)=\mu(c,d,\ell,p)\overline{\nu_\eta}(\gamma)(cz+d)^{\frac{1}{2}}\mathcal{G}_1\left(\frac{[d\ell]}{p};z\right)\quad \textit{for } \gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_0(p),$$

*where*

$$\mu(c,d,\ell,p):=\exp\left(\frac{3\pi icd\ell^2}{p^2}\right)(-1)^{\frac{c\ell}{p}}(-1)^{\lfloor\frac{d\ell}{p}\rfloor}. \tag{2.34}$$

Note that in the original paper [18], the notation for the right hand side of (2.34) was $\mu(\gamma,\ell)$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Since it only requires the value of $c$ and $d$, we modify it to $\mu(c,d,\ell,p)$ here for convenience. We use the above transformation formula to construct our $(p-1)$-dimensional multiplier system $\mu_p$.

**Definition 2.11.** *Let $p \geq 5$ be a prime. Define $M_p : \Gamma_0(p) \to \mathrm{M}_{p-1}(\mathbb{C})$ by*

$$M_p(\gamma) := \sum_{\ell=1}^{p-1} \mu(c,d,\ell,p) E_{\ell,[d\ell]} \quad \text{and} \quad \mu_p(\gamma) := \overline{\nu_\eta}(\gamma) M_p(\gamma), \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p),$$

*where $E_{j,k}$ is the $(p-1)\times(p-1)$ matrix unit whose $(j,k)$-entry equals 1 and all the other entries equal 0.*

This matrix has the following compatibility properties:

**Proposition 2.12.** *Let $p \geq 5$ be a prime and $\mu_p$ be defined as in Definition 2.11. Then for all $\gamma, \gamma_1, \gamma_2 \in \Gamma_0(p)$,*

*(1)* $\det(\mu_p(\gamma)) = \pm \overline{\nu_\eta}(\gamma)^{p-1}$;

*(2)* $\mu_p(\gamma)^{-1} = \mu_p(\gamma)^{\mathrm{H}}$, *i.e.* $\mu_p(\gamma)$ *is a unitary matrix;*

*(3)* $\mu_p(-I) = e^{-\frac{\pi i}{2}} I_{p-1}$, *where* $I_{p-1} \in \mathrm{M}_{p-1}(\mathbb{C})$ *is the identity matrix;*

*(4)* $\mu_p(\gamma_1\gamma_2) = \omega_{\frac{1}{2}}(\gamma_1,\gamma_2)\mu_p(\gamma_1)\mu_p(\gamma_2)$.

*Proof.* Since $\overline{\nu_\eta}$ is a weight $-\frac{1}{2}$ multiplier system on $\mathrm{SL}_2(\mathbb{Z})$, it suffices to prove the corresponding properties for $M_p$ in weight 1. Denote $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. When $(d,c)=1$, we have $p \nmid d$ and for $1 \leq \ell \leq p-1$, $d\ell$ runs over all residue classes modulo $p$ and vice versa. Thus, $\mu_p(\gamma)$ is an matrix with only one non-zero entry in every row and every column. Let $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ be the signature of the permutation $\sigma : \ell \to [d\ell]$ in $(\mathbb{Z}/p\mathbb{Z})^\times$. By (2.34), we have

$$\begin{aligned}
\det M_p(\gamma) &= \mathrm{sgn}(\sigma) \prod_{\ell=1}^{p-1} \mu(c,d,\ell,p) \\
&= \mathrm{sgn}(\sigma) \exp\left(\frac{3\pi i c d}{p^2} \cdot \frac{(p-1)p(2p-1)}{6}\right) (-1)^{\frac{c}{p}\cdot\frac{(p-1)p}{2}} \prod_{\ell=0}^{p-1} (-1)^{\left\lfloor \frac{d\ell}{p} \right\rfloor} \\
&= \mathrm{sgn}(\sigma)(-1)^{c(d+1)\cdot\frac{p-1}{2}} \prod_{\ell=0}^{p-1} (-1)^{d\ell + [d\ell]} \\
&= \mathrm{sgn}(\sigma)(-1)^{(c+1)(d+1)\cdot\frac{p-1}{2}} = \mathrm{sgn}(\sigma),
\end{aligned}$$

where we have used the following facts for any $x, y \in \mathbb{Z}$: $(-1)^{xp} = (-1)^x$; $\left\lfloor \frac{x}{p} \right\rfloor \equiv x + [x] \pmod 2$; if $(x,y)=1$ then $(x+1)(y+1)$ is even.

For (2), it suffices to show that $M_p(\gamma)$ is a unitary matrix. Since $P_\sigma := \sum_{\ell=1}^{p-1} E_{\ell,[d\ell]}$ is a permutation matrix with $P_\sigma^{-1} = P_\sigma^T$, we have

$$\begin{aligned}
M_p(\gamma)^{-1} &= \Big(\mathrm{diag}\{\mu(c,d,\ell,p) : 1 \leq \ell \leq p-1\} \cdot P_\sigma\Big)^{-1} \\
&= P_\sigma^T \cdot \mathrm{diag}\{\overline{\mu(c,d,\ell,p)} : 1 \leq \ell \leq p-1\} = M_p(\gamma)^{\mathrm{H}}.
\end{aligned}$$

For (3), we have $\mu(0,-1,\ell,p) = -1$ for all $\ell$ and $p$. For (4), it suffices to show that $M_p : \Gamma_0(p) \to \mathrm{GL}_{p-1}(\mathbb{C})$ is multiplicative, which was proved in [18, Theorem 4.1]. ☐

Note that Proposition 2.12 has proved all the requirements in Definition 2.6 except (4). We verify the conditions among the cusps $\infty$ and 0 of $\Gamma_0(p)$ here. Since $\mu_p$ and $\overline{\mu_p}$ will appear together in the next section, for simplicity we denote $\alpha_{+\mathfrak{a}}^{(\ell)} := \alpha_{\mu_p,\mathfrak{a}}^{(\ell)}$ and $\alpha_{-\mathfrak{a}}^{(\ell)} := \alpha_{\overline{\mu_p},\mathfrak{a}}^{(\ell)}$ for the cusp $\mathfrak{a}$. Since

$$\mu_p\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\right) = e(-\tfrac{1}{24})I_{p-1}$$

is a diagonal matrix, we have

$$\alpha_{+\infty}^{(\ell)} = \tfrac{1}{24} \quad \text{and } n_{+\infty} := n - \tfrac{1}{24} \text{ for all } n \in \mathbb{Z}. \tag{2.35}$$

For $\overline{\mu_p}$ we have $\alpha_{-\infty}^{(\ell)} = \frac{23}{24}$ and $n_{-\infty} := n - \frac{23}{24}$. Moreover, we can take the scaling matrix (see (1.10)) of the cusp 0 of $\Gamma_0(p)$ as $\sigma_0 = \left(\begin{smallmatrix}0 & -1/\sqrt{p}\\ \sqrt{p} & 0\end{smallmatrix}\right)$. Since $\sigma_0\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_0^{-1} = \left(\begin{smallmatrix}1&0\\-p&1\end{smallmatrix}\right)$ and by (1.19)

$$\nu_\eta\left(\sigma_0\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_0^{-1}\right) = i\nu_\eta\left(\left(\begin{smallmatrix}-1&0\\p&-1\end{smallmatrix}\right)\right) = i\left(\frac{-1}{p}\right)e\left(-\tfrac{5p}{24}\right) = e\left(\tfrac{p}{24}\right),$$

we have

$$\mu_p\left(\sigma_0\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)\sigma_0^{-1}\right) = \operatorname{diag}\left\{e\left(-\tfrac{3\ell^2}{2p} - \tfrac{p}{24}\right)(-1)^\ell :\ 1 \le \ell \le p-1\right\}.$$

Therefore, we define $\alpha_{+0}^{(\ell)} \in [0,1)$ such that

$$e\left(-\alpha_{+0}^{(\ell)}\right) = e\left(-\tfrac{3\ell^2}{2p} - \tfrac{p}{24}\right)(-1)^\ell \quad \text{and define } n_{+0}^{(\ell)} := n - \alpha_{+0}^{(\ell)} \text{ for } n \in \mathbb{Z}. \tag{2.36}$$

Note that $\alpha_{+0}^{(\ell)} \neq 0$ because $1 \le \ell \le p-1$ and $(p,24) = 1$.

We will need the following properties in Chapter 7 where we construct certain linear combinations of Maass-Poincaré series. For each integer $r \ge 0$, we denote

$$x_r \quad \text{as the only solution of} \quad \tfrac{3}{2}x^2 - (\tfrac{1}{2} + r)x + \tfrac{1}{24} = 0 \quad \text{in } (0, \tfrac{1}{2}). \tag{2.37}$$

The sequence $\{x_r : r \ge 0\}$ has $\frac{1}{6} = x_0 > x_1 > x_2 > \cdots > 0$. Fix the prime $p \ge 5$. For each integer $r \ge 0$ and positive integer $a$ with $(a,p) = 1$, when $x_r > \frac{1}{p}$ (otherwise the following set will be empty), we define the condition set

$$\rhd\, a, r \lhd := \left\{1 \le \ell \le p-1 : \frac{[a\ell]}{p} \in (0, x_r) \cup (1 - x_r, 1)\right\} \quad \text{and} \quad \rhd\, r \lhd := \rhd\, 1, r \lhd\,. \tag{2.38}$$

By (1.39), we observe that

$$\delta_{\ell,p,a,r} > 0 \quad \text{if and only if} \quad \ell \in \rhd\, a, r \lhd\,.$$

By (2.36), we find that $\alpha_{+0}^{(\ell)}$ is the fractional part of

$$\begin{aligned} &\tfrac{3\ell^2}{2p} - \tfrac{1+2r}{2}\ell + \tfrac{p}{24}, && \text{when } 0 < \tfrac{\ell}{p} < x_r, \\ &\tfrac{3p}{2}\left(1 - \tfrac{\ell}{p}\right)^2 - \tfrac{(1+2r)p}{2}\left(1 - \tfrac{\ell}{p}\right) + \tfrac{p}{24}, && \text{when } 1 - x_r < \tfrac{\ell}{p} < 1. \end{aligned} \tag{2.39}$$

By the definition of $\delta_{\ell,p,a,r}$ in (1.39), we observe that $\alpha_{+0}^{(\ell)}$ is the fractional part of $p\delta_{\ell,p,1,r}$ when $\delta_{\ell,p,a,r} > 0$.

Hence, for every integer $r \geq 0$, we define a special vector $\mathbf{X}_r = (X_r^{(1)}, \cdots, X_r^{(p-1)})^{\mathrm{T}} \in \mathbb{Z}^{p-1}$ such that

$$X_r^{(\ell)} = \begin{cases} \lceil -p\delta_{\ell,p,1,r} \rceil, & \text{if } \delta_{\ell,p,1,r} > 0, \text{ i.e. } \ell \in \rhd r \lhd, \\ 0, & \text{otherwise and never used.} \end{cases} \tag{2.40}$$

Then we have $X_r^{([a\ell])} = \lceil -p\delta_{\ell,p,a,r} \rceil$ by (1.39) and

$$X_{r,+0}^{(\ell)} := X_r^{(\ell)} - \alpha_{+0}^{(\ell)} = -p\delta_{\ell,p,1,r} \quad \text{when } \delta_{\ell,p,1,r} > 0, \text{ i.e. when } \ell \in \rhd r \lhd,$$

which is the "correct order" to be matched for the Maass-Poincaré series in Chapter 7.

In general, for any vector $\mathbf{m} \in \mathbb{Z}^{p-1}$, we denote $m_{+0}^{(\ell)} := m^{(\ell)} - \alpha_{+0}^{(\ell)}$. For simplicity we write $\mathbf{1} - \mathbf{m} := \sum_{\ell=1}^{\infty} (1 - m^{(\ell)}) \mathfrak{e}_\ell$ and denote $\mathbf{m} \leq 0$ if $m^{(\ell)} \leq 0$ for all $\ell$. For $\overline{\mu_p}$, we have $\alpha_{-0}^{(\ell)} = 1 - \alpha_{+0}^{(\ell)} \in (0,1)$ and define $m_{-0}^{(\ell)} = m_{-0}^{(\ell)} := m^{(\ell)} - \alpha_{-0}^{(\ell)}$. We have the property

$$(1-m)_{\pm 0}^{(\ell)} = -m_{\mp 0}^{(\ell)}. \tag{2.41}$$

We have already proved the following lemma.

**Lemma 2.13.** *We have the following $(p-1)$-dimensional multiplier systems on $\Gamma_0(p)$: $\mu_p$ of weight $\frac{1}{2}$ and $\overline{\mu_p}$ of weight $-\frac{1}{2}$, in the sense of Definition 2.6.*

In addition, by [18, Corollary 4.2], or directly by (2.34), one important property for $\mu_p$ is

$$\mu_p(\gamma) = \overline{\nu_\eta}(\gamma) I_{p-1} \quad \text{for } \gamma \in \Gamma_0(p^2) \cap \Gamma_1(p). \tag{2.42}$$

Suppose $\mathbf{F} \in \mathcal{A}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$ (see (2.24)), then for each $\ell$, we have $F^{(\ell)} \in \mathcal{A}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta})$. This fact allows us to use the notations for (scalar-valued) Maass forms in §2.2 here for vector-valued Maass forms with Petersson inner product defined in (2.25). For example, $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$ is the space of weight $\frac{1}{2}$ vector-valued square-integrable functions in $\mathcal{A}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$. For any $\mathbf{F} \in \mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$, we have $F^{(\ell)} \in \mathcal{L}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta})$. It clearly follows that $\Delta_k$ is a self-adjoint operator on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$. The spectrum of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$ is contained in the spectrum of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta})$, which includes a discrete spectrum $\frac{3}{16} = \lambda_0 \leq \lambda_1 \leq \cdots$ of finite multiplicity and a continuous spectrum $[\frac{1}{4}, \infty)$. For each eigenvalue $\lambda$ of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$, we write $\lambda = \frac{1}{4} + r^2$ for $r \in i[0, \frac{1}{4}) \cup [0, \infty)$ and call $r$ the spectral parameter. We still denote $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r)$ as the space of Maass eigenforms with spectral parameter $r$. With the property

$$\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r) \subseteq \bigoplus_{\ell=1}^{p-1} \tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}, r),$$

we have the following proposition.

**Proposition 2.14.** *The space $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r)$ is finite dimensional.*

*Proof.* For any $\mathbf{V}(z,r) = \left(V^{(1)}(z,r), \cdots, V^{(p-1)}(z,r)\right) \in \tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r)$, we have $V^{(1)}(z,r) \in \tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}, r)$, which is a finite-dimensional space. Moreover, for each $2 \leq d \leq p-1$, we have $(d,p) = 1$ and we can pick $\gamma_d = \left(\begin{smallmatrix} * & * \\ p & d \end{smallmatrix}\right) \in \Gamma_0(p)$. By the definition of $\mu_p$, we have

$$V^{(d)}(z,r) = \overline{\mu(p,d,1,p)} \nu_\eta(\gamma_d) V^{(1)}(z,r),$$

i.e. the other components are determined by the first one. We conclude with

$$\dim \mathcal{L}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r) \leq \dim \mathcal{L}_{\frac{1}{2}}(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}, r).$$

□

We summarize these properties in the following lemma for future convenience.

**Lemma 2.15.** *Suppose* $\mathbf{F} : \mathbb{H} \to \mathbb{C}^{p-1}$ *satisfies*

$$\mathbf{F}(\gamma z) = \mu_p(\gamma)(cz+d)^{\frac{1}{2}} \mathbf{F}(z) \quad \textit{for } \gamma \in \Gamma_0(p).$$

*Then for each* $\ell$, $1 \leq \ell \leq p-1$, $F^{(\ell)}$ *satisfies*

$$F^{(\ell)}(\gamma z) = \overline{\nu_\eta}(\gamma)(cz+d)^{\frac{1}{2}} F^{(\ell)}(z) \quad \textit{for } \gamma \in \Gamma_0(p^2) \cap \Gamma_1(p).$$

*If we denote* $\mathbf{G}(z) := \mathbf{F}(24z)$ *and hence* $G^{(\ell)}(z) = F^{(\ell)}(24z)$, *we have*

$$G^{(\ell)}(\gamma z) = \nu_\theta(\gamma)(cz+d)^{\frac{1}{2}} G^{(\ell)}(z) \quad \textit{for } \gamma \in \Gamma_1(576p^2).$$

*Moreover, the map* $z \to 24z$ *gives an injection*

$$\begin{array}{ccc} S_{\frac{1}{2}}\left(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}\right) & \to & S_{\frac{1}{2}}(\Gamma_1(576p^2), \nu_\theta) \\ f & \to & g \textit{ defined by } g(z) := f(24z). \end{array}$$

*Proof.* This is directly proved by our discussion above and the following fact:

$$w_{\frac{1}{2}}\left(\left(\begin{smallmatrix} a & 24b \\ c/24 & d \end{smallmatrix}\right), \left(\begin{smallmatrix} \sqrt{24} & 0 \\ 0 & 1/\sqrt{24} \end{smallmatrix}\right)\right) \overline{w_{\frac{1}{2}}\left(\left(\begin{smallmatrix} \sqrt{24} & 0 \\ 0 & 1/\sqrt{24} \end{smallmatrix}\right), \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)} = 1$$

for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(24)$. □

### 2.4.3 Vector-valued Kloosterman sums

In this subsection we define the vector-valued Kloosterman sums with $(k, \mu) = (\frac{1}{2}, \mu_p)$ or $(-\frac{1}{2}, \overline{\mu_p})$. First we consider the cusp pair $\infty\infty$. Let $m, n, c \in \mathbb{Z}$ with $p|c$. Define

$$\mathbf{S}_{\infty\infty}(m,n,c,\mu) := \sum_{\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_0(p)/\Gamma_\infty} e\left(\frac{m_{\pm\infty} a + n_{\pm\infty} d}{c}\right) \mu(\gamma)^{-1} \sum_{\ell=1}^{p-1} \frac{\mathfrak{e}_\ell}{\sin(\frac{\pi \ell}{p})}. \tag{2.43}$$

Since $\mu\left(\left(\begin{smallmatrix} a & * \\ * & d \end{smallmatrix}\right)\right)^{-1}$ maps the entry at $[a\ell]$ to $\ell$, we extract the $\ell$-th entry of the vector $\mathbf{S}_{\infty\infty}(m,n,c,\mu)$ as

$$\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu) = \sum_{\substack{d \pmod{c}^* \\ ad \equiv 1 \pmod{c}}} e\left(\frac{m_{\pm\infty} a + n_{\pm\infty} d}{c}\right) \frac{\mu(\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right))^{-1} \mathfrak{e}_{[a\ell]}}{\sin(\frac{\pi[a\ell]}{p})} =: S^{(\ell)}_{\infty\infty}(m,n,c,\nu)\mathfrak{e}_\ell. \tag{2.44}$$

For the cusp pair $0\infty$ there are more requirements for our application. For every integer $r \geq 0$, recall our definitions on $x_r$ in (2.37), $\alpha_{\pm 0}$ in (2.36), and $\rhd r \lhd$ in (2.38). For cusp pair $(\mathfrak{a}, \mathfrak{b})$, we define $\mu_{\mathfrak{ab}}(\gamma)$ for

$\gamma \in \sigma_{\mathfrak{a}}^{-1}\Gamma_0(p)\sigma_{\mathfrak{b}}$ as in [38, (3.4)]. Hence, by $\sigma_\infty = I$ and $w_k(\gamma, I) = 1$, $\mu_{0\infty}(\gamma)$ is defined for $\gamma \in \sigma_0^{-1}\Gamma_0(p)$ and given by

$$\mu_{0\infty}(\gamma) = \mu(\sigma_0\gamma) w_k(\sigma_0^{-1}, \sigma_0\gamma). \tag{2.45}$$

For every integer $r \geq 0$ and any vector $\mathbf{m} \in \mathbb{Z}^{p-1}$, we define the Kloosterman sum

$$\mathbf{S}_{0\infty}(\mathbf{m}, n, a, \mu; r) := \sum_{\substack{\gamma = \begin{pmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{pmatrix} \\ \gamma \in \Gamma_\infty \backslash \sigma_0^{-1}\Gamma_0(p)/\Gamma_\infty}} \mu_{0\infty}(\gamma)^{-1} \sum_{\ell \in \rhd r \lhd} e\left(\frac{m_{\pm 0}^{(\ell)} \frac{c}{\sqrt{p}} - n_{\pm\infty} b\sqrt{p}}{-a\sqrt{p}}\right) \mathfrak{e}_\ell. \tag{2.46}$$

Note that $\sigma_0\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$ for $\gamma \in \Gamma_\infty \setminus \sigma_0^{-1}\Gamma_0(p)/\Gamma_\infty$ in the summation above, hence $\mu_{0\infty}(\gamma)^{-1}$ maps the entry at $[a\ell]$ to $\ell$. Also note that only the values $m^{(\ell)}$ for $\ell \in\rhd a, r \lhd$ are used because $\ell \in\rhd a, r \lhd$ is equivalent to $[a\ell] \in\rhd r \lhd$. Therefore, by denoting $\gamma = \left(\begin{smallmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right)$, the $\ell$-th entry of $\mathbf{S}_{0\infty}(\mathbf{m}, n, a, \mu; r)$ is

$$\begin{aligned}\mathbf{S}_{0\infty}^{(\ell)}(m^{([a\ell])}, n, a, \mu; r) &= \begin{cases} \displaystyle\sum_{\substack{b \pmod a)^* \\ 0<c<pa,\, p|c \\ \text{s.t. } ad-bc=1}} \mu_{0\infty}(\gamma)^{-1} e\left(\frac{m_{\pm 0}^{([a\ell])} \frac{c}{\sqrt{p}} - n_{\pm\infty} b\sqrt{p}}{-a\sqrt{p}}\right) \mathfrak{e}_{[a\ell]}, & \text{if } \ell \in\rhd a, r \lhd, \\ 0\mathfrak{e}_\ell, & \text{otherwise} \end{cases} \\ &=: S_{0\infty}^{(\ell)}(m^{([a\ell])}, n, a, \mu_p; r)\mathfrak{e}_\ell. \end{aligned} \tag{2.47}$$

In Theorem 1.14, we pick $\mathbf{X}_r$ defined in (2.40) for every integer $r \geq 0$ and have

$$\mathbf{S}_{0\infty}(\mathbf{X}_r, n, a, \mu_p; r) = \sum_{\substack{\gamma = \begin{pmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{pmatrix} \\ \gamma \in \Gamma_\infty \backslash \sigma_0^{-1}\Gamma_0(p)/\Gamma_\infty}} \mu_{0\infty}(\gamma)^{-1} \sum_{\ell \in \rhd r \lhd} e\left(\frac{X_{r,+0}^{(\ell)} \frac{c}{\sqrt{p}} - n_{+\infty} b\sqrt{p}}{-a\sqrt{p}}\right) \mathfrak{e}_\ell.. \tag{2.48}$$

By denoting $\gamma = \left(\begin{smallmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right)$, we extract the $\ell$-th entry of the vector $\mathbf{S}_{0\infty}(\mathbf{X}_r, n, a, \mu_p; r)$:

$$\begin{aligned}\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p; r) &= \begin{cases} \displaystyle\sum_{\substack{b \pmod a)^* \\ 0<c<pa,\, p|c \\ \text{s.t. } ad-bc=1}} \mu_{0\infty}(\gamma)^{-1} e\left(\frac{X_{r,+0}^{([a\ell])} \frac{c}{\sqrt{p}} - n_{+\infty} b\sqrt{p}}{-a\sqrt{p}}\right) \mathfrak{e}_{[a\ell]}, & \text{if } \ell \in\rhd a, r \lhd, \\ 0\mathfrak{e}_\ell, & \text{otherwise} \end{cases} \\ &=: S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p; r)\mathfrak{e}_\ell. \end{aligned} \tag{2.49}$$

### 2.4.4 Vector-valued holomorphic modular forms

Let $\nu$ be a weight $k = \pm\frac{1}{2}$ (one-dimensional) multiplier system on the congruence subgroup $\Gamma$. Recall $M_k(\Gamma, \nu)$ as the space of weight $k$ holomorphic modular forms and $S_k(\Gamma, \nu)$ as the space of weight $k$ holomorphic cusp forms. Every $f \in M_k(\Gamma, \nu)$ satisfies the transformation property

$$f(\gamma z) = \nu(\gamma)(cz + d)^k f(z) \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma.$$

Similarly, if $\mu$ is a weight $k = \pm\frac{1}{2}$ $D$-dimensional multiplier system on $\Gamma$ (Definition 2.6), then we denote the space of weight $k$ holomorphic modular forms on $(\Gamma, \mu)$ by $M_k(\Gamma, \mu)$ and the corresponding space of cusp forms by $S_k(\Gamma, \mu)$.

From now on we take the prime $p \geq 5$ and let $(k, \mu) = (\frac{1}{2}, \mu_p)$ or $(-\frac{1}{2}, \overline{\mu_p})$ on $\Gamma_0(p)$. By Lemma 2.15 and using the fact that $\alpha_{\pm\infty} \neq 0$ in (2.35), we have

$$M_{\frac{1}{2}}(\Gamma_0(p), \mu_p) = S_{\frac{1}{2}}(\Gamma_0(p), \mu_p) \subseteq \bigoplus_{\ell=1}^{p-1} S_{\frac{1}{2}}\left(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}\right). \tag{2.50}$$

Similar as Proposition 2.14, we also have

$$\dim M_{\frac{1}{2}}(\Gamma_0(p), \mu_p) \leq \dim S_{\frac{1}{2}}\left(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}\right). \tag{2.51}$$

Lemma 2.15 also shows that, for any $f \in S_{\frac{1}{2}}\left(\Gamma_0(p^2) \cap \Gamma_1(p), \overline{\nu_\eta}\right) \subseteq S_{\frac{1}{2}}\left(\Gamma_1(p^2), \overline{\nu_\eta}\right)$, the map $z \to 24z$ gives

$$g \in S_{\frac{1}{2}}(\Gamma_1(576p^2), \nu_\theta) \quad \text{for } g(z) := f(24z). \tag{2.52}$$

The Serre-Stark basis theorem (Theorem 2.2) implies the following lemma.

**Lemma 2.16.** *Fix a prime $p \geq 5$. Let $f \in M_{\frac{1}{2}}(\Gamma_1(576p^2), \nu_\theta)$ have Fourier expansion as*

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n$$

*where $a_f(m) = 0$ for all $m \not\equiv -1 \pmod{24}$. Then if $p \not\equiv -1 \pmod{24}$, we have $f = 0$; if $p \equiv -1 \pmod{24}$, we have that $f$ is a multiple of $\eta(24pz)$.*

*Proof.* By Theorem 2.2, if $\theta_{\psi,t} \in M_{\frac{1}{2}}(\Gamma_1(576p^2), \nu_\theta)$, then whenever $\psi(n) \neq 0$, i.e. whenever $(n, r(\psi)) = 1$, we have $tn^2 \equiv -1 \pmod{24}$. Since $t | 144p^2$ and $p^2 \equiv 1 \pmod{24}$ for primes $p \geq 5$, we only have the possibility if $t = p \equiv -1 \pmod{24}$. Then we have $r(\psi) | 12$, hence $r(\psi) = 1, 3$ or $12$.

Since $\psi$ is primitive, $r(\psi) = 1$ means $\psi(n) = 1$ for all $n$, hence $\psi(2) = 1$ while $p \cdot 2^2 \not\equiv -1 \pmod{24}$. When $r(\psi) = 3$, the only primitive character is $(\frac{-3}{\cdot})$ which is odd, not to mention $\psi(2) = -1$ and $p \cdot 2^2 \not\equiv -1 \pmod{24}$.

When $r(\psi) = 12$, the only primitive character is $(\frac{12}{\cdot})$. Note that

$$\eta(24z) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2}$$

and the lemma follows. ∎

**Lemma 2.17.** *If $\mathbf{F} = \sum_{\ell=1}^{p-1} F^{(\ell)}(z)\mathfrak{e}_\ell \in M_{\frac{1}{2}}(\Gamma_0(p), \mu_p)$ has Fourier expansion*

$$F^{(\ell)}(z) = \sum_{n=1}^{\infty} a_F^{(\ell)}(n) q^{n-\frac{1}{24}} \quad \textit{for each } 1 \leq \ell \leq p-1,$$

*then $\mathbf{F} = \mathbf{0}$.*

*Proof.* Consider $\mathbf{F}(24z)$. By Lemma 2.16 and (2.52), if $p \not\equiv -1 \pmod{24}$ we already have the desired result. If $p \equiv -1 \pmod{24}$, we have $F^{(\ell)}(z) = c^{(\ell)}\eta(pz)$ for some constant $c^{(\ell)} \in \mathbb{C}$ and for each $\ell$. By [39, Corollary 3.5], $\eta(pz) \in M_{\frac{1}{2}}(\Gamma_0(p), (\frac{\cdot}{p})\overline{\nu_\eta})$ because $p \equiv -1 \pmod{24}$.

Now we take $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ p & 1 \end{smallmatrix}\right)$. By Proposition 2.10, we have

$$(pz+1)^{-\frac{1}{2}} F^{(\ell)}(\gamma z) = \mu(p,1,\ell,p)\overline{\nu_\eta(\gamma)} F^{(\ell)}(z),$$

while by $F^{(\ell)}(z) = c^{(\ell)}\eta(pz)$, we have

$$(pz+1)^{-\frac{1}{2}} F^{(\ell)}(\gamma z) = (\tfrac{1}{p})\overline{\nu_\eta(\gamma)} F^{(\ell)}(z).$$

However, $\mu(p,1,\ell,p) = \exp(\frac{3\pi i \ell^2}{p})(-1)^\ell$ cannot be $\pm 1$. Then the only possible case is $c^{(\ell)} = 0$ for all $1 \leq \ell \leq p-1$ and we have $\mathbf{F} = \mathbf{0}$. □

# Chapter 3

# Sums of Kloosterman sums: general bounds

In this section we first record the estimates by Goldfeld and Sarnak in [16]. Although their original paper did not provide a uniform bound (in $m$, $n$ and $x$), Pribitkin [34] derived a uniform bound with polynomial growth in $|\tilde{m}\tilde{n}|$. Such uniform bound is weaker than Theorem 1.7, but it works for all weight $k \in \mathbb{R}$ multiplier systems.

Nevertheless, a bound of polynomial growth in $|\tilde{m}\tilde{n}|$ is enough to ensure the convergence of certain Maass-Poincaré series. When we want to prove the exact formulas for ranks modulo $p \geq 5$, the proof requires uniform bounds for sums of vector-valued Kloosterman sums. As a generalization of [16], we prove such uniform bounds in the vector-valued case, which helps us in the proof of Theorem 1.14 in Chapter 7.

## 3.1 The work of Goldfeld and Sarnak

In this section we briefly outline the work of Goldfeld and Sarnak [16] restricted to half-integral weight. Let $\Gamma$ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$. Let $\nu$ be a weight $k \in \mathbb{Z} + \frac{1}{2}$ multiplier system on $\Gamma$. Recall the Kloosterman sums defined in (1.12). We define the Kloosterman-Selberg zeta function as

$$Z_{m,n,\nu}(s) := \sum_{\substack{c>0,\ \text{s.t.} \\ \left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \Gamma}} \frac{S(m,n,c,\nu)}{c^{2s}}. \tag{3.1}$$

We will omit the condition $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \Gamma$ for simplicity. When $\Gamma = \Gamma_0(N)$ for some positive integer $N$, we will write the sum simply as $N|c > 0$. In this section, we keep the integers $m, n > 0$.

Goldfeld and Sarnak proved the following theorem.

**Theorem 3.1** ([16, Theorem 1])**.** *The function $Z_{m,n,\nu}(s)$ is meromorphic in $\operatorname{Re} s > \frac{1}{2}$ with at most a finite number of simple poles in $(\frac{1}{2}, 1)$, and satisfies the growth condition*

$$Z_{m,n,\nu}(s) \ll_{\Gamma,m,n,\nu,k} \frac{|s|^{\frac{1}{2}}}{\sigma - \frac{1}{2}}$$

*for $s = \sigma + it$, $\sigma > \frac{1}{2}$ and $t \to \infty$.*

The following Mellin transform was from [16, (2.3)] and will be used frequently in the next section. For $\alpha > 0$ and $\mathrm{Re}(s + \frac{1}{2} \pm \mu) > 0$, we have

$$\int_0^\infty e^{-2\pi\alpha y} y^s W_{\beta,\mu}(4\pi\alpha y)\frac{dy}{y} = (4\pi\alpha)^{-s}\frac{\Gamma(s+\frac{1}{2}+\mu)\Gamma(s+\frac{1}{2}-\mu)}{\Gamma(s-\beta+1)}. \tag{3.2}$$

For a positive integer $m$, we define the non-holomorphic Poincaré series as

$$P_m(z;s,\nu) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\nu(\gamma)} j(\gamma,z)^{-k} e(\tilde{m}\gamma z)(\mathrm{Im}\, y)^s. \tag{3.3}$$

The above series converge absolutely and uniformly in any compact subset of $\mathrm{Re}\, s > 1$. Moreover, $P_m(z;s,\nu) \in \mathcal{L}_k(\Gamma,\nu)$ (see §2.2). They also satisfy the recursion relation

$$P_m(z;s,\nu) = -4\pi\tilde{m}(s-\tfrac{k}{2})\mathscr{R}_{s(1-s)}(P_m(z;s+1,\nu)), \tag{3.4}$$

where $\mathscr{R}_\lambda := (\Delta_k + \lambda)^{-1}$ is the resolvent of $\Delta_k$. As $\Delta_k$ may have exceptional eigenvalues at $\lambda_j < \frac{1}{4}$, the resolvent $\mathscr{R}_{s(1-s)}$ is holomorphic in $\mathrm{Re}\, s > \frac{1}{2}$ except possible points at $s_j$ such that $s_j(1-s_j) = \lambda_j$. Hence by (3.4), $P_m(z;s,\nu)$ can be meromorphically continued to $\mathrm{Re}\, s > \frac{1}{2}$ except a finite number of simple poles at such $s_j$.

Let $u_j(z) \in \mathcal{L}_k(\Gamma,\nu)$ be the eigenfunction of $\Delta_k$ with eigenvalue $\lambda_j = s_j(1-s_j)$. Then $u_j$ has Fourier expansion

$$u_j(z) = \rho_j(0,y) + \sum_{\substack{n\in\mathbb{Z}\\ \tilde{n}\neq 0}} \rho_j(n) W_{\frac{k}{2}\,\mathrm{sgn}\,\tilde{n},\, s_j-\frac{1}{2}}(4\pi|\tilde{n}|y)\, e(\tilde{n}x), \tag{3.5}$$

where $W_{\beta,\mu}(y)$ is the Whittaker function, $\rho_j(0,y) = \rho_j(0)y^{s_j} + \rho_j'(0)y^{1-s_j}$ if $n = \alpha_\nu = 0$, and $\rho_j(0,y) = 0$ otherwise. The residues of $P_m(z;s,\nu)$ at $s = s_j \in (\frac{1}{2},1)$ can be computed as in [16, (2.5)]:

$$\operatorname*{Res}_{s=s_j} P_m(z;s,\nu) = \overline{\rho_j(m)} 4\pi\tilde{m}^{1-s_j}\frac{\Gamma(2s_j-1)}{\Gamma(s_j-\frac{k}{2})}u_j(z). \tag{3.6}$$

Theorem 3.1 follows from the following two lemmas and

$$\left|\frac{\Gamma(2s+1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2}+2)}\right| \sim \frac{|t|^{-\frac{1}{2}}}{\sqrt{2}\pi}, \quad \text{for } |t|\to\infty.$$

**Lemma 3.2** ([16, Lemma 2])**.** *Let $s = \sigma + it$ where $\frac{1}{2} < \sigma \le 2$ and $|t| > 1$. Then*

$$\int_{\Gamma\backslash\mathbb{H}} |P_m(z;s,\nu)|^2 \frac{dxdy}{y}^2 \ll_{\Gamma,\nu,k} \frac{m^2}{(\sigma-\frac{1}{2})^2}.$$

**Lemma 3.3** ([16, Lemma 3])**.** *For $m,n>0$, $\sigma > \frac{1}{2}$, we have*

$$\int_{\Gamma\backslash\mathbb{H}} P_m(z;s,\nu)\overline{P_n(z;\overline{s}+2,\nu)}\frac{dxdy}{y^2} = e(-\tfrac{k}{4})4^{-s-1}\pi^{-1}\tilde{n}^{-2}\frac{\Gamma(2s+1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2}+2)}Z_{m,n,\nu}(s) + R_{m,n,\nu}(s),$$

*where $R_{m,n,\nu}(s)$ is holomorphic in $\sigma > \frac{1}{2}$ and $|R_{m,n,\nu}(s)| \ll_{\Gamma,m,n,\nu,k} \frac{1}{\sigma-\frac{1}{2}}$ in this region.*

*Remark.* Pribitkin proved a bound for $R_{m,n,\nu}(s)$ which is uniform in $m$ and $n$ in [34, Lemma 2].

With the help of Theorem 3.1, using an argument as in the proof of the prime number theorem (e.g. [40, Chapter 18]), Goldfeld and Sarnak proved:

**Theorem 3.4** ([16, Theorem 2]). *Let*

$$\beta = \limsup_{c\to\infty} \frac{\log|S(m,n,c,\nu)|}{\log c},$$

*then*

$$\sum_{c\leq x} \frac{S(m,n,c,\nu)}{c} = \sum_{s_j\in(\frac{1}{2},1)} \tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} + O_{\Gamma,m,n,\nu,k}\left(x^{\frac{\beta}{3}+\varepsilon}\right),$$

*where $\tau_j(m,n)$ are defined as in Theorem 1.7.*

In the next section, we generalize this bound to the vector-valued Kloosterman sums, which helps us in Chapter 7.

## 3.2 Sums of vector-valued Kloosterman sums

In this section, we follow [16] to prove an asymptotic formula for sums of certain vector-valued Kloosterman sums. Let $p\geq 5$ be a prime number. Let $(k,\mu)$ be either $(\frac{1}{2},\mu_p)$ or $(-\frac{1}{2},\overline{\mu_p})$. For $n\in\mathbb{Z}$ and $\mathbf{m}\in\mathbb{Z}^{p-1}$, recall the notations $\alpha_{\pm\infty}$, $n_{\pm\infty}$, $\alpha_{\pm0}^{(\ell)}$ and $m_{\pm0}^{(\ell)}$ for $1\leq\ell\leq p-1$ introduced before Lemma 2.13 and recall the Kloosterman sums defined in (2.43) and (2.46).

By Proposition 2.14, for every spectral parameter $r_j$ of $\lambda_j=\frac{1}{4}+r_j^2$ in the discrete spectrum, we can pick an orthonormal (under the inner product (2.25)) basis of $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p),\mu_p,r_j)$ denoted as $\mathrm{OB}(r_j)$. For every $\mathbf{V}(z;r_j)\in\mathrm{OB}(r_j)$, we have $V^{(\ell)}(z;r_j)\in\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta},r_j)$ which has the Fourier expansion

$$V^{(\ell)}(z;r_j) = c_\infty^{(\ell)}(0,y) + \sum_{\substack{n\in\mathbb{Z}\\ n_{+\infty}\neq0}} \rho_{j,\infty}^{(\ell)}(n) W_{\frac{k}{2}\operatorname{sgn} n_{+\infty},ir_j}(4\pi|n_{+\infty}|y)e(n_{+\infty}x) \tag{3.7}$$

as in (3.5). Since $\alpha_{+\infty}\neq0$, we have $c_\infty^{(\ell)}(0,y)=0$. The Fourier expansion of $\mathbf{V}(z;r_j)$ at the cusp 0 is given by

$$(V^{(\ell)}|_k\sigma_0)(z;r_j) = c_0^{(\ell)}(0,y) + \sum_{\substack{n\in\mathbb{Z}\\ n_{+0}^{(\ell)}\neq0}} \rho_{j,0}^{(\ell)}(n) W_{\frac{1}{4}\operatorname{sgn} n,\,ir_j}\left(4\pi|n_{+0}^{(\ell)}|y\right)e(n_{+0}^{(\ell)}x). \tag{3.8}$$

Here $c_0^{(\ell)}(0,y)=0$ because $\alpha_{+0}^{(\ell)}\neq0$ for all $1\leq\ell\leq p-1$.

Specially, for the case of $r_0=\frac{i}{4}$, by the proof of Proposition 2.14, any $\mathbf{V}(z;r_0)\in\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p),\mu_p,r_0)$ satisfies $\mathbf{V}^{(\ell)}(z;r_0)\in\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta},r_0)$. From (2.13), there exists a one-to-one correspondence between $\mathbf{V}(z;r_0)\in\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p),\mu_p,r_0)$ and $F\in M_{\frac{1}{2}}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta})$ by

$$V^{(1)}(z;r_0) = y^{\frac{1}{4}}F(z). \tag{3.9}$$

By (2.14), for the Fourier expansion of $\mathbf{V}(z;r_0)$, since $\alpha_{+\infty}^{(\ell)}>0$ (2.35) and $\alpha_{+0}^{(\ell)}>0$ (2.36), we have

$$\begin{aligned} V^{(\ell)}(z;r_0) &= \sum_{n_{+\infty}>0} \rho_{0,\infty}^{(\ell)}(n) W_{\frac14,ir_0}(4\pi n_{+\infty} y) e(n_{+\infty}x), \\ (V^{(\ell)}|_k\sigma_0)(z;r_0) &= \sum_{n_{+0}^{(\ell)}>0} \rho_{0,0}^{(\ell)}(n) W_{\frac14,ir_0}\left(4\pi n_{+0}^{(\ell)} y\right) e(n_{+0}^{(\ell)}x), \end{aligned} \tag{3.10}$$

i.e. $\rho_{0,\infty}^{(\ell)}(n)=\rho_{0,0}^{(\ell)}(n)=0$ if $n\le 0$.

We will prove the following theorem in this section.

**Theorem 3.5.** *Fix an integer $r\ge 0$, a prime $p\ge 5$, and let $1\le L,\ell\le p-1$. For $m\in\mathbb{Z}$ with $m\le 0$, $\mathbf{m}\in\mathbb{Z}^{p-1}$ with $\mathbf{m}\le 0$, and $M=\max_{\ell\in\rhd r\lhd}\{|m_{+0}^{(\ell)}|\}$, we have $m_{+\infty}<0$ (see (2.35)), $m_{+0}^{(\ell)}<0$ for all $\ell$ (see (2.36)), and the following results:*

$$\sum_{c\le x:\, p|c} \frac{S_{\infty\infty}^{(\ell)}(m,n,c,\mu_p)}{c} = \sum_{\frac12<s_j\le\frac34} \tau_{j,\infty}^{(\ell)}(m,n)\frac{x^{2s_j-1}}{2s_j-1} + O_{p,\varepsilon}\left(|mn|^3 x^{\frac13+\varepsilon}\right), \tag{3.11}$$

$$\sum_{\substack{a\le x:\\ p\nmid a,\, [\overline{a\ell}]=L}} \frac{S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p;r)}{a\sqrt{p}} = \sum_{\frac12<s_j\le\frac34} \tau_{j,0,(L)}^{(\ell)}(m^{(L)},n)\frac{x^{2s_j-1}}{2s_j-1} + O_{p,\varepsilon}\left(|m_{+0}^{(L)}n|^3 x^{\frac13+\varepsilon}\right), \tag{3.12}$$

$$\sum_{\substack{a\le x:\\ p\nmid a,\, [\overline{a\ell}]\in\rhd r\lhd}} \frac{S_{0\infty}^{(\ell)}(m^{([\overline{a\ell}])},n,a,\mu_p;r)}{a\sqrt{p}} = \sum_{\frac12<s_j\le\frac34} \tau_{j,0}^{(\ell)}(\mathbf{m},n)\frac{x^{2s_j-1}}{2s_j-1} + O_{p,\varepsilon}\left(|Mn|^3 x^{\frac13+\varepsilon}\right), \tag{3.13}$$

*where*

$$\tau_{j,\infty}^{(\ell)}(m,n) = e(\tfrac18)\left(\sum_{L=1}^{p-1}\frac{\overline{\rho_{j,\infty}^{(L)}(m)}}{\sin(\frac{\pi L}{p})}\right)\frac{\rho_{j,\infty}^{(\ell)}(n)}{\sin(\frac{\pi\ell}{p})}\cdot\frac{\Gamma(s_j+\frac14\operatorname{sgn} n_{+\infty})\Gamma(2s_j-1)}{\pi^{2s_j-1}|4m_{+\infty}n_{+\infty}|^{s_j-1}\Gamma(s_j-\frac14)},$$

$$\tau_{j,0,(L)}^{(\ell)}(m^{(L)},n) = e(-\tfrac18)\overline{\rho_{j,0}^{(L)}(m^{(L)})}\cdot\frac{\rho_{j,\infty}^{(\ell)}(n)}{\sin(\frac{\pi\ell}{p})}\cdot\frac{\Gamma(s_j+\frac14\operatorname{sgn} n_{+\infty})\Gamma(2s_j-1)}{\pi^{2s_j-1}|4m_{+0}^{(L)}n_{+\infty}|^{s_j-1}\Gamma(s_j-\frac14)},$$

*and $\tau_{j,0}^{(\ell)}(\mathbf{m},n)=\sum_{L\in\rhd r\lhd}\tau_{j,0,(L)}^{(\ell)}(m^{(L)},n)$. Here all the sums on $s_j$ run over the exceptional eigenvalues $\lambda_j=s_j(1-s_j)\in[\frac{3}{16},\frac14)$ of $\Delta_{\frac12}$ on $\mathcal{L}_{\frac12}(\Gamma_0(p),\mu_p)$. The coefficients $\rho_{j,\infty}^{(\ell)}(n)$ and $\rho_{j,0}^{(\ell)}(m^{(\ell)})$ are the Fourier coefficients of an eigenform $\mathbf{V}(z;r_j)$ of $\Delta_{\frac12}$ in a orthonormal basis $\mathrm{OB}(r_j)$ of $\tilde{\mathcal{L}}_{\frac12}(\Gamma_0(p),\mu_p,r_j)$, defined in (3.7) and (3.8). The summation term corresponding to a single $s_j$ should be understood as the sum over all $\mathbf{V}(z;r_j)\in\mathrm{OB}(r_j)$.*

*Remark.* Here we have an important clarification of our notation. The notation $r$ always means the integer $r\ge0$ which appears in the Kloosterman sum $S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p;r)$, in $x_r$ defined in (2.37), and in $\rhd a,r\lhd$ defined in (2.38). The notation $r_j$, with subscript $j\ge0$, is the spectral parameter of the eigenvalue $\lambda_j=\frac14+r_j^2$ of $\Delta_{\frac12}$ on $(\Gamma_0(p),\mu_p)$.

**Corollary 3.6.** *With the same setting as Theorem 3.5, there exists a $\delta>0$ such that for all $1\le L,\ell\le p-1$,* *we have*

$$\sum_{c\le x:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} \ll_p |m_{+\infty}n_{+\infty}|^3 x^{\frac12-\delta}$$
$$\sum_{\substack{a\le x:\\ p\nmid a,\,[a\ell]=L}} \frac{S^{(\ell)}_{0\infty}(m^{(L)},n,a,\mu_p;r)}{a\sqrt{p}} \ll_p |m^{(L)}_{+0}n_{+\infty}|^3 x^{\frac12-\delta},$$
$$\sum_{\substack{a\le x:\\ p\nmid a,\,[a\ell]\in\rhd r\lhd}} \frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p;r)}{a\sqrt{p}} \ll_p |Mn_{+\infty}|^3 x^{\frac12-\delta}.$$

*Proof of Corollary 3.6.* Granted Theorem 3.5, it suffices to determine the growth rate of $\rho^{(\ell)}_{j,0}(n)$ and $\rho^{(\ell)}_{j,0}(m^{(\ell)})$. By the discussion in Proposition 2.14, these coefficients are also the Fourier coefficients of an eigenform of $\Delta_{\frac12}$ in a orthonormal basis of $\tilde{\mathcal{L}}_{\frac12}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta},r_j)$. Then we can get the growth rate from the spaces of scalar valued Maass eigenforms of $\Delta_{\frac12}$.

For $\frac{3}{16}=\lambda_0=s_0(1-s_0)$, i.e. $s_0=\frac34$, we know that $\tilde{\mathcal{L}}_{\frac12}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta},r_j)$ corresponds to holomorphic modular forms $M_{\frac12}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta})$ as in (3.9). Since we have $m_{+\infty}<0$ and $m^{(\ell)}_{+0}<0$, by (3.9), we get $\rho^{(\ell)}_{0,\infty}(m)=0$ and $\rho^{(\ell)}_{0,0}(m^{(\ell)})=0$. Thus, the term $x^{2s_0-1}=x^{\frac12}$ for $s_0=\frac34$ is not contained in each sum.

Since the exceptional eigenvalues of $\Delta_{\frac12}$ on $\mathcal{L}_{\frac12}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta},r_j)$ are discrete, there exists $\delta>0$ such that $2s_j-1<\frac12-\delta$ for all $s_j\in(\frac12,\frac34)$. For these $j$, both $\rho^{(\ell)}_{j,\infty}(n)$ and $\rho^{(\ell)}_{j,0}(m^{(\ell)})$ are $O_p(1)$. The corollary follows. □

We generalize the method in [16] (to the vector-valued setting) to prove Theorem 3.5. Define the Kloosterman-Selberg zeta functions as

$$Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s) := \sum_{c\le x:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c^{2s}}, \quad Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s) := \sum_{\substack{a\le x:\, p\nmid a,\\ [a\ell]=L}} \frac{S^{(\ell)}_{0\infty}(m^{(L)},n,a,\mu_p;r)}{(a\sqrt{p})^{2s}}. \tag{3.14}$$

and $Z^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s) := \sum_{L\in\rhd r\lhd} Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)$. Recall the remark following Theorem 3.5 for the integer $r\ge0$ involved in the zeta functions above.

We address the proof of (3.11) in the next subsection. The proof of (3.13) is in the subsequent subsection.

### 3.2.1 The cusp $\infty$

Recall the notation $(k,\mu)=(\frac12,\mu_p)$ or $(-\frac12,\overline{\mu_p})$. For $m>0$, we define the weight $k$ non-holomorphic vector-valued Poincaré series as

$$\mathbf{U}(z;s,k,m,\mu) := \sum_{\ell=1}^{p-1}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mu(\gamma)^{-1}j(\gamma,z)^{-k}\frac{y^s}{|cz+d|^{2s}}\frac{e(m_{\pm\infty}\gamma z)\mathfrak{e}_\ell}{\sin(\frac{\pi\ell}{p})} \tag{3.15}$$

where $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and the $\pm$ in $m_{\pm\infty}$ is chosen depending on the sign of $k$. We also denote

$$\mathbf{U}_{(\ell)}(z;s,k,m,\mu) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mu(\gamma)^{-1}j(\gamma,z)^{-k}\frac{y^s}{|cz+d|^{2s}}\frac{e(m_{\pm\infty}\gamma z)}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell \tag{3.16}$$

as the part of the Poincaré series generated by $\mathfrak{e}_\ell$.

*Remark.* Note the difference between $\mathbf{U}_{(\ell)}(z;s,k,m,\mu)$ and $\mathbf{U}^{(\ell)}(z;s,k,m,\mu)$: $\mathbf{U}_{(\ell)}$ is defined only by the terms associated to $\mathfrak{e}_\ell$, while $\mathbf{U}^{(\ell)}$ is the $\ell$-th entry of the vector $\mathbf{U}$. All the components of $\mathbf{U}^{(\ell)}$ for $L: 1\le L\neq \ell\le p-1$ are zero, but this is not true for $\mathbf{U}_{(\ell)}$.

The Poincaré series $\mathbf{U}(z;s,k,m,\mu)$ and $\mathbf{U}_{(\ell)}(z;s,k,m,\mu)$ converge absolutely and uniformly in any compact subset of $\operatorname{Re}s>1$ and are in $\mathcal{L}_k(\Gamma_0(p),\mu)$. To show the transformation law, e.g.

$$\mathbf{U}(\gamma_1 z;s,k,m,\mu)=\mu(\gamma_1)j(\gamma_1,z)^k\mathbf{U}(z;s,k,m,\mu)\quad\text{for }\gamma_1\in\Gamma_0(p),$$

it suffices to show that for $\gamma_1,\gamma_2\in\Gamma_0(p)$, we have

$$\begin{aligned}\mu(\gamma_2\gamma_1^{-1})^{-1}j(\gamma_2\gamma_1^{-1},\gamma_1 z)^{-k}&=\big(\mu(\gamma_2)\mu(\gamma_1^{-1})\big)^{-1}\overline{w_k(\gamma_2,\gamma_1^{-1})}j(\gamma_2\gamma_1^{-1},\gamma_1 z)^{-k}\\&=\mu(\gamma_1)\mu(\gamma_2)^{-1}j(\gamma_2,z)^{-k}j(\gamma_1^{-1},\gamma_1 z)^{-k}j(\gamma_2\gamma_1^{-1},\gamma_1 z)^{k}j(\gamma_2\gamma_1^{-1},\gamma_1 z)^{-k}\\&=\mu(\gamma_1)j(\gamma_1,z)^k\cdot\mu(\gamma_2)^{-1}j(\gamma_2,z)^{-k},\end{aligned}\tag{3.17}$$

where we have used this trick: since $w_k(\gamma,\gamma')$ does not depend on $z\in\mathbb{H}$, we have

$$w_k(\gamma,\gamma')=j(\gamma',\gamma''z)^k j(\gamma,\gamma'\gamma''z)^k j(\gamma\gamma',\gamma''z)^{-k}\quad\text{for all }\gamma''\in\mathrm{SL}_2(\mathbb{Z}),\tag{3.18}$$

as well as the properties $\mu(\gamma_1^{-1})=\mu(\gamma_1)^{-1}$ and $j(\gamma_1^{-1},\gamma_1 z)^k j(\gamma_1,z)^k=1$.

For $\operatorname{Re}s>1$, we can compute the Fourier expansion of $\mathbf{U}(z;s,k,m,\mu)$ in the same way as the scalar-valued case. The contribution from $c=0$ equals

$$\sum_{\ell=1}^{p-1}\frac{y^s}{\sin(\frac{\pi\ell}{p})}e(m_{\pm\infty}z)\mathfrak{e}_\ell.$$

When $c>0$, the contribution from a single $c$ equals

$$\begin{aligned}&\sum_{\ell=1}^{p-1}\sum_{\substack{d\ (\mathrm{mod}\ c)^*\\ ad\equiv1\ (\mathrm{mod}\ c)}}\sum_{t\in\mathbb{Z}}\mu\big(\big(\begin{smallmatrix}a&b+ta\\c&d+tc\end{smallmatrix}\big)\big)^{-1}\left(\frac{cz+d+tc}{|cz+d+tc|}\right)^{-k}\frac{y^s}{|cz+d+tc|^{2s}}\frac{e\left(m_{\pm\infty}\frac{az+b+ta}{cz+d+tc}\right)}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell\\&=\sum_{\ell=1}^{p-1}\sum_{\substack{d\ (\mathrm{mod}\ c)^*\\ ad\equiv1\ (\mathrm{mod}\ c)}}\frac{\mu\big(\big(\begin{smallmatrix}a&*\\c&d\end{smallmatrix}\big)\big)^{-1}}{c^{2s}}e\left(\frac{m_{\pm\infty}a}{c}\right)\sum_{t\in\mathbb{Z}}\left(\frac{z+\frac dc+t}{|z+\frac dc+t|}\right)^{-k}\frac{e(t\alpha_{\pm\infty})y^s}{|z+\frac dc+t|^{2s}}\frac{e\left(-\frac{m_{\pm\infty}}{c^2(z+\frac dc+t)}\right)\mathfrak{e}_\ell}{\sin(\frac{\pi\ell}{p})},\end{aligned}$$

where we have used $\frac{az+b}{cz+d}=\frac ac-\frac{1}{c(cz+d)}$, Definition 2.11, and the property (1.9) for $\nu_\eta$. If we denote

$$f_\pm(z):=\sum_{t\in\mathbb{Z}}\left(\frac{z+t}{|z+t|}\right)^{-k}\frac{e(t\alpha_{\pm\infty})\,y^s}{|z+t|^{2s}}e\left(-\frac{m_{\pm\infty}}{c^2(z+t)}\right),$$

then $f_\pm(z)e(\alpha_{\pm\infty}x)$ have period 1 and we get the following Fourier expansion by Poisson summation:

$$f_\pm(z)=\sum_{n_{\pm\infty}>0}\frac{B(c,m_{\pm\infty},t_{\pm\infty},y,s,k)}{\sin(\frac{\pi\ell}{c})}e(n_{\pm\infty}x)\quad\text{and}\quad f_\pm\left(z+\frac dc\right)=e\left(\frac{n_{\pm\infty}d}{c}\right)f_\pm(z),$$

where (with the substitution $x = yu$)

$$B(c, m_{\pm\infty}, t_{\pm\infty}, y, s, k) := y \int_{\mathbb{R}} \left(\frac{u+i}{|u+i|}\right)^{-k} e\left(\frac{-m_{\pm\infty}}{c^2 y(u+i)} - t_{\pm\infty} yu\right) \frac{du}{y^{2s}(u^2+1)^s}. \tag{3.19}$$

Therefore, the Fourier expansion of $\mathbf{U}(z; s, k, m, \mu)$ at the cusp $\infty$ is

$$\begin{aligned}\mathbf{U}(z; s, k, m, \mu) = \sum_{\ell=1}^{p-1} &\frac{y^s e(m_{\pm\infty} z)\mathfrak{e}_\ell}{\sin(\frac{\pi\ell}{p})} \\ &+ y^s \sum_{t\in\mathbb{Z}} \sum_{p|c>0} \frac{\mathbf{S}_{\infty\infty}(m, t, c, \mu)}{c^{2s}} B(c, m_{\pm\infty}, t_{\pm\infty}, y, s, k) e(t_{\pm\infty} x),\end{aligned} \tag{3.20}$$

where $\mathbf{S}_{\infty\infty}(m, n, c, \mu)$ is defined in (2.43).

Moreover, for $\operatorname{Re} s > 1$ and $\lambda = s(1-s)$, we still have the recursion relation

$$\mathbf{U}(z; s, k, m, \mu) = -4\pi m_{\pm\infty}(s - \tfrac{k}{2})\mathscr{R}_\lambda \left(\mathbf{U}(z; s+1, k, m, \mu)\right), \tag{3.21}$$

$$\mathbf{U}_{(\ell)}(z; s, k, m, \mu) = -4\pi m_{\pm\infty}(s - \tfrac{k}{2})\mathscr{R}_\lambda \left(\mathbf{U}_{(\ell)}(z; s+1, k, m, \mu)\right), \tag{3.22}$$

where $\mathscr{R}_\lambda = (\Delta_k + \lambda)^{-1}$ is the resolvent of $\Delta_k$. Since $\frac{1}{2} < \operatorname{Re} s < 1$ implies $\lambda < \frac{1}{4}$, we know that $\mathbf{U}(z; s, k, m, \mu)$ and every $\mathbf{U}_{(\ell)}(z; s, k, m, \mu)$ can be meromorphically continued to the half plane $\operatorname{Re} s > \frac{1}{2}$ with a finite number of simple poles at $s_j$ for $\frac{1}{2} < s_j < \frac{3}{4}$.

Recall that we choose $m_{+\infty} < 0$ in the condition for Theorem 3.5, hence $m \le 0$ and $(1-m)_{-\infty} = -m_{+\infty} > 0$. Also recall that $\operatorname{OB}(r_j)$ is a orthonormal basis of $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r_j)$ and let $\mathbf{V}(z; r_j) \in \operatorname{OB}(r_j)$ (see (3.7)). The residue of $\overline{\mathbf{U}(\cdot; \overline{s}, -\frac{1}{2}, 1-m, \overline{\mu_p})}$ at $s = s_j$ is then given by

$$\sum_{\mathbf{V}\in \operatorname{OB}(r_j)} \operatorname*{Res}_{s=s_j} \left\langle \overline{\mathbf{U}(\cdot; \overline{s}, -\tfrac{1}{2}, 1-m, \overline{\mu_p})}, \mathbf{V}(\cdot; r_j)\right\rangle \mathbf{V}(z; r_j).$$

We are going to compute the inner product (defined by (2.25)) by applying the Mellin transform (3.2). Note that $W_{\frac{k}{2}\operatorname{sgn} m_{+\infty}, ir_j}(4\pi|m_{+\infty}|y) \in \mathbb{R}$ and $s_j = \frac{1}{2} + ir_j$ for $r_j \in i(0, \frac{1}{4})$. We get

$$\begin{aligned}&\left\langle \overline{\mathbf{U}(\cdot; \overline{s}, -\tfrac{1}{2}, 1-m, \overline{\mu_p})}, \mathbf{V}(\cdot; r_j)\right\rangle \\ &= \int_{\Gamma_0(p)\backslash\mathbb{H}} \mathbf{V}(z; r_j)^{\mathrm{H}} \left(\overline{\sum_{\ell=1}^{p-1} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \overline{\mu_p}(\gamma)^{-1} j(\gamma, z)^{\frac{1}{2}} (\operatorname{Im}\gamma z)^{\overline{s}} \frac{e((1-m)_{-\infty}\gamma z)}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell}\right) \frac{dxdy}{y^2} \\ &= \sum_{\ell=1}^{p-1} \int_{\Gamma_\infty\backslash\mathbb{H}} \mathbf{V}(z; r_j)^{\mathrm{H}} y^s e(m_{+\infty} x) e^{2\pi m_{+\infty} y} \frac{\mathfrak{e}_\ell}{\sin(\frac{\pi\ell}{p})} \frac{dxdy}{y^2} \\ &= \sum_{\ell=1}^{p-1} \frac{\overline{\rho_\infty^{(\ell)}(m)}}{\sin(\frac{\pi\ell}{p})} \int_0^\infty e^{-2\pi|m_{+\infty}|y} W_{-\frac{1}{4}, ir_j}(4\pi|m_{+\infty}|y) y^{s-1} \frac{dy}{y} \\ &= (4\pi|m_{+\infty}|)^{1-s} \frac{\Gamma(s-s_j)\Gamma(s+s_j-1)}{\Gamma(s+\frac{1}{4})} \sum_{\ell=1}^{p-1} \frac{\overline{\rho_\infty^{(\ell)}(m)}}{\sin(\frac{\pi\ell}{p})}.\end{aligned}$$

The residue of $\overline{\mathbf{U}(\cdot;\overline{s},-\frac{1}{2},1-m,\overline{\mu_p})}$ at $s=s_j$ is then

$$\sum_{\mathbf{V}\in\mathrm{OB}(r_j)}\operatorname*{Res}_{s=s_j}\left\langle\overline{\mathbf{U}(\cdot;\overline{s},-\tfrac{1}{2},1-m,\overline{\mu_p})},\mathbf{V}(\cdot;r_j)\right\rangle\mathbf{V}(z;r_j)$$
$$=\sum_{\mathbf{V}\in\mathrm{OB}(r_j)}(4\pi|m_{+\infty}|)^{1-s_j}\frac{\Gamma(2s_j-1)}{\Gamma(s_j+\frac{1}{4})}\sum_{\ell=1}^{p-1}\frac{\overline{\rho_\infty^{(\ell)}(m)}}{\sin(\frac{\pi\ell}{p})}\mathbf{V}(z;r_j). \tag{3.23}$$

The following lemma still holds as [16, Lemma 1] because the proofs are essentially the same except for the difference of scalar-valued Petersson inner product and the vector-valued one in (2.25). We omit the proof here.

**Lemma 3.7.** *Let $s=\sigma+it$, $(k,\mu)=(\frac{1}{2},\mu_p)$ or $(-\frac{1}{2},\overline{\mu_p})$ and $m>0$. For $\frac{1}{2}<\sigma\leq 2$ and $|t|>1$ we have*

$$\langle\mathbf{U}(\cdot;s,k,m,\mu),\mathbf{U}(\cdot;s,k,m,\mu)\rangle\ll_p\frac{m^2}{(\sigma-\frac{1}{2})^2},$$
$$\left\langle\mathbf{U}_{(\ell)}(\cdot;s,k,m,\mu),\mathbf{U}_{(\ell)}(\cdot;s,k,m,\mu)\right\rangle\ll_p\frac{m^2}{(\sigma-\frac{1}{2})^2}.$$

The following useful equation follows from [41, (3.384.9)]: for $y>0$, $\beta\neq 0$, $k=\pm\frac{1}{2}$, $\operatorname{Re}s>\frac{1}{2}$, we have

$$\int_{\mathbb{R}}\frac{(x+i)^{-k}}{(x^2+1)^{s-\frac{k}{2}}}e^{-2\pi i\beta xy}dx=\frac{e(-\frac{k}{4})\pi(\pi|\beta|y)^{s-1}}{\Gamma(s+\frac{k}{2}\operatorname{sgn}\beta)}W_{\frac{k}{2}\operatorname{sgn}\beta,\,\frac{1}{2}-s}(4\pi|\beta|y). \tag{3.24}$$

In the next lemma we compute the inner product of two Poincaré series. Recall the definition in (3.14).

**Lemma 3.8.** *Suppose that $m_{+\infty}<0$ and let $1\leq\ell\leq p-1$. Then*

$$Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s)\quad\textit{is meromorphic in }\operatorname{Re}s>\tfrac{1}{2}$$

*with at most a finite number of simple poles in $(\frac{1}{2},1)$. Moreover, when $\operatorname{Re}s>\frac{1}{2}$, for $n_{+\infty}>0$ we have*

$$\left\langle\overline{\mathbf{U}(\cdot;\overline{s},-\tfrac{1}{2},1-m,\overline{\mu_p})},\mathbf{U}_{(\ell)}(\cdot;\overline{s}+2,\tfrac{1}{2},n,\mu_p)\right\rangle$$
$$=\frac{e(-\frac{1}{8})}{4^{s+1}\pi n_{+\infty}^2}\cdot\frac{\Gamma(2s+1)}{\Gamma(s+\frac{1}{4})\Gamma(s+\frac{7}{4})}Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s)+O_p\left(\frac{|m_{+\infty}n_{+\infty}|}{\sigma-\frac{1}{2}}\right),$$

*and for $n_{+\infty}<0$ we have*

$$\left\langle\overline{\mathbf{U}(\cdot;\overline{s},-\tfrac{1}{2},1-m,\overline{\mu_p})},\ \overline{\mathbf{U}_{(\ell)}(\cdot;s+2,-\tfrac{1}{2},1-n,\overline{\mu_p})}\right\rangle$$
$$=\frac{e(-\frac{1}{8})}{4^{s+1}\pi|n_{+\infty}|^2}\cdot\frac{\Gamma(2s+1)}{\Gamma(s-\frac{1}{4})\Gamma(s+\frac{9}{4})}Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s)+O_p\left(\frac{|m_{+\infty}n_{+\infty}|}{\sigma-\frac{1}{2}}\right).$$

*Proof.* We compute similarly as [16, Lemma 2] with the following properties: $\mu_p(\gamma)$ is unitary, $\mu_p(\gamma^{-1})=\mu_p(\gamma)^{-1}$, and $j(\gamma^{-1},z)^{\frac{1}{2}}j(\gamma,\gamma^{-1}z)^{\frac{1}{2}}=1$. For the first equality in the following computation, we write $\overline{\mathbf{U}}$ to denote the whole first term in the inner product. Suppose that $\operatorname{Re} s_2 > \operatorname{Re} s_1 > \frac{1}{2}$. When $n_{+\infty} > 0$, we have

$$\begin{aligned}
&\left\langle \overline{\mathbf{U}(\cdot;\overline{s}_1,-\tfrac{1}{2},1-m,\overline{\mu_p})}, \mathbf{U}_{(\ell)}(\cdot;\overline{s_2},\tfrac{1}{2},n,\mu_p) \right\rangle \\
&= \int_{\Gamma_0(p)\backslash\mathbb{H}} \left( \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mu_p(\gamma)^{-1} j(\gamma,z)^{-\frac{1}{2}} (\operatorname{Im}\gamma z)^{\overline{s_2}} \frac{e(n_{+\infty}\gamma z)}{\sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell \right)^{\mathrm{H}} \overline{\mathbf{U}}\, \frac{dxdy}{y^2} \\
&= \int_{\Gamma_0(p)\backslash\mathbb{H}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} j(\gamma,\gamma^{-1}z)^{\frac{1}{2}} y^{s_2} e(-n_{+\infty}x) \frac{e^{-2\pi n_{+\infty}y}}{\sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell^{\mathrm{T}} \mu_p(\gamma) \\
&\qquad \cdot \overline{\mu_p}(\gamma^{-1}) j(\gamma^{-1},z)^{\frac{1}{2}} \overline{\mathbf{U}(z;\overline{s_1},-\tfrac{1}{2},1-m,\overline{\mu_p})} \frac{dxdy}{y^2} \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} y^{s_2} e(-n_{+\infty}x) \frac{e^{-2\pi n_{+\infty}y}}{\sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell^{\mathrm{T}}\, \overline{\mathbf{U}(z;\overline{s_1},-\tfrac{1}{2},1-m,\overline{\mu_p})} \frac{dxdy}{y^2}.
\end{aligned}$$

Then we use the Fourier expansion (3.20) with (2.44) to continue:

$$\begin{aligned}
&= \sum_{p|c>0} \frac{S_{\infty\infty}^{(\ell)}(m,n,c,\mu_p)}{c^{2s_1}} \int_0^\infty y^{s_2-s_1} e^{-2\pi n_{+\infty}y}\, \overline{B(c,(1-m)_{-\infty},(1-n)_{-\infty},y,\overline{s_1},-\tfrac{1}{2})} \frac{dy}{y} \\
&= Z_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1) \int_0^\infty \int_{\mathbb{R}} \frac{y^{s_2-s_1} e^{-2\pi n_{+\infty}y}}{(u^2+1)^{s_1}} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}} e(-n_{+\infty}uy) \frac{dxdy}{y} + R_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1,s_2) \\
&= \frac{e(-\frac{1}{8})}{4^{s+1}\pi n_{+\infty}^2} \cdot \frac{\Gamma(2s_1+1)}{\Gamma(s_1+\frac{1}{4})\Gamma(s_1+\frac{7}{4})} Z_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1) + R_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1,s_2),
\end{aligned}$$

where

$$\begin{aligned}
R_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1,s_2) &= Z_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1) \int_0^\infty \int_{\mathbb{R}} \frac{y^{s_2-s_1} e^{-2\pi n_{+\infty}y}}{(u^2+1)^{s_1}} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}} \\
&\qquad \cdot e(-n_{+\infty}uy) \left( e\left( \frac{-m_{+\infty}(u+i)}{c^2 y(u^2+1)} \right) - 1 \right) \frac{dxdy}{y} \\
&= Z_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1) O\left( \frac{|m_{+\infty}n_{+\infty}|}{c^2(\sigma_1-\frac{1}{2})} \right) = O\left( \frac{|m_{+\infty}n_{+\infty}|}{\sigma_1-\frac{1}{2}} \right),
\end{aligned}$$

which is holomorphic when $\sigma_1 > \frac{1}{2}$. The lemma follows by setting $s_2 = s_1 + 2$.

Similarly, when $n_{+\infty} < 0$ we have

$$\begin{aligned}
&\left\langle \overline{\mathbf{U}(\cdot;\overline{s}_1,-\tfrac{1}{2},1-m,\overline{\mu_p})},\ \overline{\mathbf{U}_{(\ell)}(\cdot;s_2,-\tfrac{1}{2},1-n,\overline{\mu_p})} \right\rangle \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} y^{s_2} e(-n_{+\infty}x) \frac{e^{2\pi n_{+\infty}y}}{\sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell^{\mathrm{T}} \overline{\mathbf{U}(z;\overline{s_1},-\tfrac{1}{2},1-m,\overline{\mu_p})} \frac{dxdy}{y^2} \\
&= \frac{e(-\frac{1}{8})\pi^{1+s_1-s_2}}{4^{s_2-1}|n_{+\infty}|^{s_2-s_1}} \cdot \frac{\Gamma(s_2-s_1)\Gamma(s_2+s_1-1)}{\Gamma(s_1-\frac{1}{4})\Gamma(s_2+\frac{1}{4})} Z_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1) + R_{\substack{\infty\infty\\ m,n,+}}^{(\ell)}(s_1,s_2)
\end{aligned}$$

where

$$R^{(\ell)}_{\substack{\infty\infty\\ m,n,\mu_p}}(s_1,s_2) = \sum_{p|c>0} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c^{2s_1}} \int_0^\infty \int_{\mathbb{R}} \frac{y^{s_2-s_1} e^{2\pi n_{+\infty} y}}{(u^2+1)^{s_1}} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}}$$

$$\cdot\, e(-n_{+\infty} uy) \left( e\left(\frac{-m_{+\infty}(u+i)}{c^2 y(u^2+1)}\right) - 1\right) \frac{dxdy}{y}$$

$$= \sum_{p|c>0} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c^{2s_1}} O\left(\frac{|m_{+\infty} n_{+\infty}|}{c^2(\sigma_1-\frac{1}{2})}\right) = O\left(\frac{|m_{+\infty} n_{+\infty}|}{\sigma_1 - \frac{1}{2}}\right).$$

The lemma follows by setting $s_2 = \overline{s_1} + 2$. □

Combining Lemma 3.7 and Lemma 3.8, we have the following proposition.

**Proposition 3.9.** *For $m_{+\infty} < 0$, $s = \sigma + it$, $\sigma > \frac{1}{2}$ and $|t| \to \infty$, we have*

$$Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s) \ll_p \frac{|m_{+\infty} n_{+\infty}|^3 |s|^{\frac{1}{2}}}{\sigma - \frac{1}{2}}.$$

Now we can prove the first case of Theorem 3.5.

*Proof of* (3.11) *in Theorem 3.5.* Denote $s = \sigma + it$. Fix any $\varepsilon > 0$, by Proposition 3.9, $Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s) \ll_{p,\varepsilon} \zeta(1+\varepsilon)$ for $\sigma = 1+\varepsilon$, and the Phragmén-Lindelöf principle, we have

$$Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(\tfrac{1+s}{2}) \ll_{p,\varepsilon} |m_{+\infty} n_{+\infty}|^3 |t|^{\frac{1}{2} - \frac{\sigma}{2} + \varepsilon} \quad \text{for } 0 < \varepsilon \le \sigma \le 1 + \varepsilon. \tag{3.25}$$

Then following the argument of [16, §3], by the proof of the prime number theorem as in [40, Chapter 18], we have

$$\sum_{p|c\le x} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} = \frac{1}{2\pi i} \int_{1-iT}^{1+iT} Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(\tfrac{1+s}{2}) \frac{x^s}{s} ds + O\left( |m_{+\infty} n_{+\infty}|^3 \frac{x^{1+\varepsilon}}{T}\right).$$

By Lemma 3.8, the function $Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(\frac{1+s}{2})$ has at most a finite number of simple poles at $2s_j - 1 \in (0,1)$ (note that we are using $\frac{1+s}{2}$ rather than $s$), where $\lambda_j = s_j(1-s_j) < \frac{1}{4}$ are the discrete eigenvalues of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$. Shifting the line of integration above to $\operatorname{Re} s = \varepsilon$ such that $2s_j - 1 > \varepsilon$ for all $\lambda_j < \frac{1}{4}$, with the help of (3.25) we obtain

$$\sum_{p|c\le x} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} = \sum_{\frac{1}{2} < s_j < \frac{3}{4}} \operatorname*{Res}_{s=2s_j-1} Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(\tfrac{1+s}{2}) \frac{x^{2s_j-1}}{2s_j-1} + O(|m_{+\infty} n_{+\infty}|^3 x^{\frac{1}{3}+\varepsilon}),$$

where we have chosen $T = x^{\frac{2}{3}}$.

For the residue, by Lemma 3.8, it suffices to compute the residue of the two inner products in that lemma. Let $\mathrm{OB}(r_j)$ be an orthonormal basis of $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$ and let $\mathbf{V}(z;r_j) \in \mathrm{OB}(r_j)$. When $n_{+\infty} > 0$, combining

Lemma 3.8 and (3.23), we are left to compute the following inner product using (3.2):

$$\begin{aligned}\left\langle \mathbf{V}(\cdot;r_j),\mathbf{U}_{(\ell)}(\cdot;\overline{s_j}+2,\tfrac12,n,\mu_p)\right\rangle &= \int_{\Gamma_\infty\backslash\Gamma_0(p)} y^{s_j+2}e(-n_{+\infty}x)\frac{e^{-2\pi n_{+\infty}y}\mathfrak{e}_\ell^{\mathrm{T}}}{\sin(\frac{\pi\ell}{p})}\mathbf{V}(z;r_j)\frac{dxdy}{y^2}\\ &= (4\pi n_{+\infty})^{-s_j-1}\frac{\rho_{j,\infty}^{(\ell)}(n)}{\sin(\frac{\pi\ell}{p})}\frac{\Gamma(2s_j+1)}{\Gamma(s_j+\frac74)}.\end{aligned} \tag{3.26}$$

Similarly, when $n_{+\infty}<0$ we compute:

$$\begin{aligned}\left\langle \mathbf{V}(\cdot;r_j),\overline{\mathbf{U}_{(\ell)}(\cdot;s_j+2,-\tfrac12,1-n,\overline{\mu_p})}\right\rangle &= \int_{\Gamma_\infty\backslash\Gamma_0(p)} y^{s_j+2}e(-n_{+\infty}x)\frac{e^{2\pi n_{+\infty}y}\mathfrak{e}_\ell^{\mathrm{T}}}{\sin(\frac{\pi\ell}{p})}\mathbf{V}(z;r_j)\frac{dxdy}{y^2}\\ &= (4\pi|n_{+\infty}|)^{-s_j-1}\frac{\rho_{j,\infty}^{(\ell)}(n)}{\sin(\frac{\pi\ell}{p})}\frac{\Gamma(2s_j+1)}{\Gamma(s_j+\frac94)}.\end{aligned} \tag{3.27}$$

Combining Lemma 3.8 with (3.23), (3.26) and (3.27), we get

$$\operatorname*{Res}_{s=s_j} Z^{(\ell)}_{\substack{\infty\infty\\ m,n,+}}(s)=\sum_{\mathbf{V}\in\mathrm{OB}(r_j)}e(\tfrac18)\left(\sum_{L=1}^{p-1}\frac{\overline{\rho_{j,\infty}^{(L)}(m)}}{\sin(\frac{\pi L}{p})}\right)\frac{\rho_{j,\infty}^{(\ell)}(n)}{\sin(\frac{\pi\ell}{p})}\cdot\frac{\Gamma(s_j+\frac14\operatorname{sgn}n_{+\infty})\Gamma(2s_j-1)}{\pi^{2s_j-1}|4m_{+\infty}n_{+\infty}|^{s_j-1}\Gamma(s_j-\frac14)} \tag{3.28}$$

and finish the proof. □

### 3.2.2 The cusp 0

Recall from (1.10) that $\Gamma_0$ is the stabilizer of the cusp 0 in $\Gamma_0(p)$ and $\sigma_0=\left(\begin{smallmatrix}0&-1/\sqrt p\\ \sqrt p&0\end{smallmatrix}\right)$ is the scaling matrix of the cusp 0. They satisfy the property $\sigma_0^{-1}\Gamma_0\sigma_0=\Gamma_\infty$, where $\Gamma_\infty=\{\pm\left(\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right):n\in\mathbb{Z}\}$ is the stabilizer of the cusp $\infty$.

Fix an integer $r\geq 0$. Recall the definition of $x_r$ in (2.37). For $(k,\mu)=(\frac12,\mu_p)$ or $(-\frac12,\overline{\mu_p})$, we take $\mathbf{m}=(m^{(1)},\cdots,m^{(p-1)})^{\mathrm{T}}\in\mathbb{Z}^{p-1}$ for each $\ell$. Recall the definition of $\alpha_{\pm0}^{(\ell)}$ and $m_{\pm0}^{(\ell)}$ in (2.36) and of $\rhd r\lhd$ in (2.38). We write $\mathbf{m}>0$ (resp. $\mathbf{m}\leq 0$) if every entry $m^{(\ell)}>0$ (resp. every entry $m^{(\ell)}\leq 0$).

For $\mathbf{m}>0$, we define

$$\mathbf{U}_{0,(\ell)}(z;s,k,m^{(\ell)},\mu,r):=\begin{cases}\displaystyle\sum_{\gamma\in\Gamma_0\backslash\Gamma_0(p)}\mu(\gamma)^{-1}\overline{w_k(\sigma_0^{-1},\gamma)}j(\sigma_0^{-1}\gamma,z)^{-k}(\operatorname{Im}\sigma_0^{-1}\gamma z)^s e\left(m_{\pm0}^{(\ell)}\sigma_0^{-1}\gamma z\right)\mathfrak{e}_\ell, & \text{if } \ell\in\rhd r\lhd,\\ 0\mathfrak{e}_\ell, & \text{otherwise, never used.}\end{cases} \tag{3.29}$$

We also define

$$\mathbf{U}_0(z;s,k,\mathbf{m},\mu,r):=\sum_{\ell\in\rhd r\lhd}\mathbf{U}_{0,(\ell)}(z;s,k,m^{(\ell)},\mu,r). \tag{3.30}$$

Note that $\mathbf{U}_{0,(\ell)}(z;s,k,m^{(\ell)},\mu,r)$ is different from $\mathbf{U}_0^{(\ell)}(z;s,k,\mathbf{m},\mu,r)$, where $\mathbf{U}_0^{(\ell)}(z;s,k,\mathbf{m},\mu,r)$ is defined to have the same $\ell$-th entry as $\mathbf{U}_0(z;s,k,\mathbf{m},\mu,r)$ but has 0 in the other entries. The Poincaré series $\mathbf{U}_0(z;s,k,\mathbf{m},\mu,r)$ and every $\mathbf{U}_{0,(\ell)}(z;s,k,m^{(\ell)},\mu,r)$ converge absolutely and uniformly in any compact subset of $\operatorname{Re}s>1$, and are in $\mathcal{L}_k(\Gamma_0(p),\mu)$. To show the transformation law, e.g.

$$\mathbf{U}_0(\gamma_1 z;s,k,\mathbf{m},\mu,r)=\mu(\gamma_1)j(\gamma_1,z)^k\mathbf{U}_0(z;s,k,\mathbf{m},\mu,r)\quad\text{for }\gamma_1\in\Gamma_0(p),$$

it suffices to show that for $\gamma_1, \gamma_2 \in \Gamma_0(p)$, we have

$$\begin{aligned}
&\mu(\gamma_2\gamma_1^{-1})^{-1} \cdot \overline{w_k(\sigma_0^{-1}, \gamma_2\gamma_1^{-1})} j(\sigma_0^{-1}\gamma_2\gamma_1^{-1}, \gamma_1 z)^{-k} \\
&= \big(\mu(\gamma_2)\mu(\gamma_1^{-1})\big)^{-1} \overline{w_k(\gamma_2, \gamma_1^{-1})} \\
&\quad \cdot j(\sigma_0^{-1}, \gamma_2 z)^{-k} j(\gamma_2\gamma_1^{-1}, \gamma_1 z)^{-k} j(\sigma_0^{-1}\gamma_2\gamma_1^{-1}, \gamma_1 z)^{k} j(\sigma_0^{-1}\gamma_2\gamma_1^{-1}, \gamma_1 z)^{-k} \\
&= \mu(\gamma_1)\mu(\gamma_2)^{-1} j(\gamma_2, z)^{-k} j(\gamma_1^{-1}, \gamma_1 z)^{-k} j(\gamma_2\gamma_1^{-1}, \gamma_1 z)^{k} \cdot j(\sigma_0^{-1}, \gamma_2 z)^{-k} j(\gamma_2\gamma_1^{-1}, \gamma_1 z)^{-k} \\
&= \mu(\gamma_1) j(\gamma_1, z)^k \cdot \mu(\gamma_2)^{-1} j(\sigma_0^{-1}, \gamma_2 z)^{-k} j(\gamma_2, z)^{-k} \\
&= \mu(\gamma_1) j(\gamma_1, z)^k \cdot \mu(\gamma_2)^{-1} \overline{w_k(\sigma_0^{-1}, \gamma_2)} j(\sigma_0^{-1}\gamma_2, z)^{-k},
\end{aligned} \tag{3.31}$$

where we have used the trick in (3.18), $\mu(\gamma_1^{-1}) = \mu(\gamma_1)^{-1}$ and $j(\gamma_1^{-1}, \gamma_1 z)^k j(\gamma_1, z)^k = 1$.

Recall the scaling matrices $\sigma_0 = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$ and $\sigma_\infty = I$. We have the following double coset decomposition by [38, (2.32)]:

$$\sigma_0^{-1}\Gamma_0(p)\sigma_\infty = \sigma_0^{-1}\Gamma_0(p) = \bigcup_{\substack{a>0 \\ p\nmid a}} \bigcup_{b \pmod{a}^*} \Gamma_\infty \begin{pmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{pmatrix} \Gamma_\infty. \tag{3.32}$$

Then $\gamma_1 \in \Gamma_0 \setminus \Gamma_0(p) \Leftrightarrow \gamma_2 = \sigma_0^{-1}\gamma_1 \in \Gamma_\infty \setminus \sigma_0^{-1}\Gamma_0(p)$ and all choices of $\gamma_2$ can be described as

$$\gamma_2 \in \{\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) : \ a > 0, \ p \nmid a, \ b \ (\mathrm{mod}\ a)^*, \ t \in \mathbb{Z}\}.$$

Recall (2.45), Definition 2.11, (2.35) and the property (1.9) for $\nu_\eta$. We have

$$\mu_{0\infty}(\gamma) = \mu(\sigma_0\gamma) w_k(\sigma_0^{-1}, \sigma_0\gamma) \quad \text{and} \quad \mu_{0\infty}(\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)) = \mu_{0\infty}(\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) e(-t\alpha_{\pm\infty}).$$

We also have

$$w_k(\sigma_0^{-1}, \gamma) j(\sigma_0^{-1}\gamma, z)^k = j(\gamma, z)^k j(\sigma_0^{-1}, \gamma z)^k = e(-\tfrac{k}{2})(\tfrac{az+b}{|az+b|})^k \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), a > 0 \text{ and } c > 0.$$

To compute the Fourier expansion of $\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r)$ when $\operatorname{Re} s > 1$, we have

$$\begin{aligned}
&\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r) \\
&= \sum_{\ell \in \rhd r \lhd} \sum_{\substack{\gamma = \left(\begin{smallmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right) \\ \gamma \in \Gamma_\infty \setminus \sigma_0^{-1}\Gamma_0(p)}} \mu_{0\infty}(\gamma)^{-1} \left(\frac{-az-b}{|-az-b|}\right)^{-k} \frac{y^s e(m_{\pm 0}^{(\ell)}\gamma z)}{|a\sqrt{p}z + b\sqrt{p}|^{2s}} \mathfrak{e}_\ell \\
&= \sum_{\ell \in \rhd r \lhd} \sum_{\substack{a>0 \\ p\nmid a}} \sum_{\substack{b(a)^* \\ 0<c<pa,\, p|c \\ ad-bc=1}} \frac{\mu_{0\infty}\left(\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)^{-1}}{p^s} \sum_{t \in \mathbb{Z}} e(t\alpha_{\pm\infty}) \left(\frac{-z - \frac{b}{a} - t}{|-z - \frac{b}{a} - t|}\right)^{-k} \frac{y^s e\left(\frac{m_{\pm 0}^{(\ell)}(cz+d)}{-paz-pb}\right)}{|az + b + ta|^{2s}} \mathfrak{e}_\ell.
\end{aligned}$$

Note that $\frac{cz+d}{-paz-pb} = -\frac{c}{pa} - \frac{1}{pa(az+b)}$ and $\mu(c, d+tc, \ell, p) = \mu(c, d, \ell, p)$ by (2.34) for all $\ell$ and $t$. Recall that the matrix $\mu_{0\infty}\left(\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)^{-1}$ maps the entry at $[a\ell]$ to $\ell$ and $\nu_\eta\left(\left(\begin{smallmatrix} a & b+ta \\ c & d+tc \end{smallmatrix}\right)\right) = \nu_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) e(-\frac{t}{24})$. Then when

$\ell \notin \rhd a, r \lhd$, the contribution from a single $a$ to the $\ell$-th entry is zero; when $\ell \in \rhd a, r \lhd$, such contribution is

$$\frac{e(-\frac{k}{2})}{p^s a^{2s}} \sum_{\substack{b \pmod a)^* \\ 0<c<pa,\ p|c \\ \text{s.t. } ad-bc=1}} \mu_{0\infty}\left(\sigma_0^{-1}\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\right)^{-1} e\left(-\frac{m_{\pm 0}^{([a\ell])} c}{pa}\right) \mathfrak{e}_{[a\ell]}$$
$$\cdot \sum_{t\in\mathbb{Z}} e(t\alpha_{\pm\infty}) \left(\frac{z+\frac{b}{a}+t}{|z+\frac{b}{a}+t|}\right)^{-k} \frac{y^s}{|z+\frac{b}{a}+t|^{2s}} e\left(\frac{-m_{\pm 0}^{([a\ell])}}{pa^2(z+\frac{b}{a}+t)}\right)$$
$$= y^s e(-\tfrac{k}{2}) \sum_{t\in\mathbb{Z}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(m^{([a\ell])}, t, a, \mu; r)}{(a\sqrt{p})^{2s}} B(a\sqrt{p}, m_{\pm 0}^{([a\ell])}, t_{\pm\infty}, y, s, k) e(t_{\pm\infty} x).$$

Here we get the last step in the same way as the steps before (3.19). The Fourier expansion of $\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r)$ for $\operatorname{Re} s > 1$ is then

$$\begin{aligned} &\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r) \\ &= y^s e(-\tfrac{k}{2}) \sum_{\substack{a>0 \\ p\nmid a}} \sum_{\ell \in \rhd a, r\lhd} \sum_{t\in\mathbb{Z}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(m^{([a\ell])}, t, a, \mu; r)}{(a\sqrt{p})^{2s}} B(a\sqrt{p}, m_{\pm 0}^{([a\ell])}, t_{\pm\infty}, y, s, k) e(t_{\pm\infty} x) \end{aligned} \tag{3.33}$$

where $\mathbf{S}_{0\infty}^{(\ell)}(m^{([a\ell])}, t, c, \mu; r)$ is defined in (2.47).

Similarly, for $L \in \rhd r \lhd$, we can compute the Fourier expansion of $\mathbf{U}_{0,(L)}(z; s, k, m^{(L)}, \mu, r)$ (3.29) and get

$$\begin{aligned} &\mathbf{U}_{0,(L)}^{(\ell)}(z; s, k, m^{(L)}, \mu, r) \\ &= y^s e(-\tfrac{k}{2}) \sum_{\substack{a>0: \\ p\nmid a,\ [a\ell]=L}} \sum_{t\in\mathbb{Z}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(m^{(L)}, t, a, \mu; r)}{(a\sqrt{p})^{2s}} B(a\sqrt{p}, m_{\pm 0}^{(L)}, t_{\pm\infty}, y, s, k) e(t_{\pm\infty} x). \end{aligned} \tag{3.34}$$

If $\operatorname{Re} s > 1$ and $\lambda = s(1-s)$ is an eigenvalue of $\Delta_k$, we have the recurrence relation

$$\mathbf{U}_{0,(\ell)}(z; s, k, m^{(\ell)}, \mu, r) = -4\pi m_{\pm 0}^{(\ell)} \mathscr{R}_\lambda \mathbf{U}_{0,(\ell)}(z; s+1, k, m^{(\ell)}, \mu, r) \tag{3.35}$$
$$\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r) = -4\pi \sum_{\ell \in \rhd r \lhd} m_{\pm 0}^{(\ell)} \mathscr{R}_\lambda \mathbf{U}_{0,(\ell)}(z; s+1, k, m^{(\ell)}, \mu, r) \tag{3.36}$$

for $\mathscr{R}_\lambda = (\Delta_k + \lambda)^{-1}$. Then $\mathbf{U}_0(z; s, k, \mathbf{m}, \mu, r)$ and every $\mathbf{U}_{0,(\ell)}(z; s, k, m^{(\ell)}, \mu, r)$ can be meromorphically continued to the half plane $\operatorname{Re} s > \frac{1}{2}$ except at most a finite number of simple poles at $s = s_j$ with $\frac{1}{2} < s_j < 1$.

Let $\operatorname{OB}(r_j)$ be an orthonormal basis of $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p), \mu_p, r_j)$ and let $\mathbf{V}(z; r_j) \in \operatorname{OB}(r_j)$ (see (3.8)). For $\mathbf{m} \le 0$ (then $\mathbf{1} - \mathbf{m} > 0$), the residue of $\overline{\mathbf{U}_0(z; \overline{s}, -\frac{1}{2}, \mathbf{1}-\mathbf{m}, \overline{\mu_p}, r)}$ at $s = s_j$ is given by

$$\sum_{\mathbf{V} \in \operatorname{OB}(r_j)} \operatorname*{Res}_{s=s_j} \left\langle \overline{\mathbf{U}_0(\cdot; \overline{s}, -\tfrac{1}{2}, \mathbf{1}-\mathbf{m}, \overline{\mu_p}, r)}, \mathbf{V}(\cdot; r_j) \right\rangle \mathbf{V}(z; r_j). \tag{3.37}$$

We will compute the inner product below and will finally get (3.40). Recall (3.29) for the definition of $\mathbf{U}_{0,(\ell)}$.

When $\ell \in \rhd r \lhd$, we use $\mathbf{V}$ to abbreviate the second term in the inner product $\mathbf{V}(\cdot; r_j)$ and get

$$\begin{aligned}
&\left\langle \overline{\mathbf{U}_{0,(\ell)}(\cdot; \overline{s}, -\tfrac{1}{2}, \mathbf{1}-\mathbf{m}, \overline{\mu_p}, r)}, \mathbf{V}(\cdot; r_j)\right\rangle \\
&= \int_{\Gamma_0(p)\backslash\mathbb{H}} \mathbf{V}^{\mathrm{H}} \sum_{\gamma\in\Gamma_0\backslash\Gamma_0(p)} \mu_p(\gamma)^{-1} w_{-\frac{1}{2}}(\sigma_0^{-1}, \gamma) j(\sigma_0^{-1}\gamma, z)^{-\frac{1}{2}} \operatorname{Im}(\sigma_0^{-1}\gamma z)^s \overline{e((1-m)_{-0}^{(\ell)} z)} \mathfrak{e}_\ell \frac{dxdy}{y^2} \\
&= \sum_{\gamma\in\Gamma_0\backslash\Gamma_0(p)} \int_{\sigma_0^{-1}\gamma(\Gamma_0(p)\backslash\mathbb{H})} \mathbf{V}(\sigma_0 z; r_j)^{\mathrm{H}} j(\sigma_0, z)^{\frac{1}{2}} y^s e(m_{+0}^{(\ell)} x) e^{2\pi m_{+0}^{(\ell)} y} \mathfrak{e}_\ell \frac{dxdy}{y^2} \\
&= \int_0^\infty y^s e^{2\pi m_{+0}^{(\ell)} y} \frac{dy}{y^2} \int_{\mathbb{R}} (\mathbf{V}|_{\frac{1}{2}}\sigma_0)(z; r_j)^{\mathrm{H}} e(m_{+0}^{(\ell)} x) \mathfrak{e}_\ell dx,
\end{aligned}$$

where we have used the following properties: $\mu_p(\gamma)$ is unitary, $j(M, z)j(M^{-1}, Mz) = 1$ for $M \in \mathrm{SL}_2(\mathbb{R})$, and the trick in (3.18):

$$\begin{aligned}
&j(\gamma^{-1}, \sigma_0 z)^{-\frac{1}{2}} w_{-\frac{1}{2}}(\sigma_0^{-1}, \gamma) j(\sigma_0^{-1}\gamma, \gamma^{-1}\sigma_0 z)^{-\frac{1}{2}} \\
&= j(\gamma^{-1}, \sigma_0 z)^{-\frac{1}{2}} j(\gamma, \gamma^{-1}\sigma_0 z)^{-\frac{1}{2}} j(\sigma_0^{-1}, \gamma\gamma^{-1}\sigma_0 z)^{-\frac{1}{2}} j(\sigma_0^{-1}\gamma, \gamma^{-1}\sigma_0 z)^{\frac{1}{2}} j(\sigma_0^{-1}\gamma, \gamma^{-1}\sigma_0 z)^{-\frac{1}{2}} \\
&= j(\sigma_0^{-1}, \sigma_0 z)^{-\frac{1}{2}} = j(\sigma_0, z)^{\frac{1}{2}}.
\end{aligned}$$

We have also used the property that for every cusp $\mathfrak{a}$ of $\Gamma$,

$$\bigcup_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \sigma_{\mathfrak{a}}^{-1}\gamma(\Gamma\setminus\mathbb{H}) = \Gamma_\infty \setminus \mathbb{H}, \qquad \text{up to a zero-measure set.}$$

Then we can apply (3.2) to get

$$\begin{aligned}
\left\langle \overline{\mathbf{U}_{0,(\ell)}(\cdot; \overline{s}, -\tfrac{1}{2}, 1-m^{(\ell)}, \overline{\mu_p}, r)}, \mathbf{V}(\cdot; r_j)\right\rangle &= \overline{\rho_{j,0}^{(\ell)}(m^{(\ell)})} \int_0^\infty y^{s-1} e^{2\pi m_{+0}^{(\ell)} y} W_{-\frac{1}{4}, ir_j}\left(4\pi|m_{+0}^{(\ell)}|y\right) \frac{dy}{y} \\
&= \overline{\rho_{j,0}^{(\ell)}(m^{(\ell)})} (4\pi|m_{+0}^{(\ell)}|)^{1-s} \frac{\Gamma(s-s_j)\Gamma(s+s_j-1)}{\Gamma(s+\frac{1}{4})}.
\end{aligned} \tag{3.38}$$

The residue of $\overline{\mathbf{U}_{0,(\ell)}(z; \overline{s}, -\frac{1}{2}, 1-m^{(\ell)}, \overline{\mu_p}, r)}$ at $s = s_j$ is then given by the following sum combining (3.38):

$$\sum_{\mathbf{V}\in\mathrm{OB}(r_j)} \operatorname*{Res}_{s=s_j} \left\langle \overline{\mathbf{U}_{0,(\ell)}(\cdot; \overline{s}, -\tfrac{1}{2}, 1-m^{(\ell)}, \overline{\mu_p}, r)}, \mathbf{V}(\cdot; r_j)\right\rangle \mathbf{V}(z; r_j). \tag{3.39}$$

Summing up for $\ell \in \rhd r \lhd$, we get

$$\left\langle \overline{\mathbf{U}_0(z; \overline{s}, -\tfrac{1}{2}, \mathbf{1}-\mathbf{m}, \overline{\mu_p}, r)}, \mathbf{V}(\cdot; r_j)\right\rangle = \sum_{\ell\in\rhd r\lhd} \overline{\rho_{j,0}^{(\ell)}(m^{(\ell)})} (4\pi|m_{+0}^{(\ell)}|)^{1-s} \frac{\Gamma(s-s_j)\Gamma(s+s_j-1)}{\Gamma(s+\frac{1}{4})}. \tag{3.40}$$

The following lemma still holds as in [16, Lemma 1] because the proofs are essentially the same except for the difference of the scalar-valued Petersson inner product and the vector-valued one in (2.25). We omit the proof here. Recall (3.30), (3.29) and (3.16).

**Lemma 3.10.** *Let $s=\sigma+it$ with $\frac{1}{2}<\sigma\le 2$, $|t|>1$, and $M=\max_{\ell\in\rhd r\lhd}|m_{+0}^{(\ell)}|$. We have*

$$\langle \mathbf{U}_0(\cdot;s,k,\mathbf{m},\mu,r),\mathbf{U}_0(\cdot;s,k,\mathbf{m},\mu,r)\rangle \ll_p \frac{M^2}{(\sigma-\frac{1}{2})^2},$$
$$\left\langle \mathbf{U}_{0,(\ell)}(\cdot;s,k,m^{(\ell)},\mu,r),\mathbf{U}_{0,(\ell)}(\cdot;s,k,m^{(\ell)},\mu,r)\right\rangle \ll_p \frac{|m_{+0}^{(\ell)}|^2}{(\sigma-\frac{1}{2})^2}.$$

Recall (3.14) for our Kloosterman-Selberg zeta functions. We have the following lemma.

**Lemma 3.11.** *Let $\mathbf{m}\le 0$, $1\le \ell, L\le p-1$, and $\operatorname{Re}s=\sigma>\frac{1}{2}$. Then*

$$Z^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s) \quad \text{and} \quad Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s) \quad \text{are meromorphic in } \operatorname{Re}s>\tfrac{1}{2}$$

*with at most a finite number of simple poles in $(\frac{1}{2},1)$. Moreover, when $n_{+\infty}>0$, we have*

$$\begin{aligned}&\left\langle \overline{\mathbf{U}_{0,(L)}(\cdot;\overline{s},-\tfrac{1}{2},1-m^{(L)},\overline{\mu_p},r)},\ \mathbf{U}_{(\ell)}(\cdot;\overline{s}+2,\tfrac{1}{2},n,\mu_p,r)\right\rangle\\ &\quad=\frac{e(\frac{1}{8})\Gamma(2s+1)}{4^{s+1}\pi n_{+\infty}^2\Gamma(s+\frac{1}{4})\Gamma(s+\frac{7}{4})}Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)+R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s),\end{aligned}$$

$$\begin{aligned}&\left\langle \overline{\mathbf{U}_{0}(\cdot;\overline{s},-\tfrac{1}{2},\mathbf{1}-\mathbf{m},\overline{\mu_p},r)},\ \mathbf{U}_{(\ell)}(\cdot;\overline{s}+2,\tfrac{1}{2},n,\mu_p,r)\right\rangle\\ &\quad=\frac{e(\frac{1}{8})\Gamma(2s+1)}{4^{s+1}\pi n_{+\infty}^2\Gamma(s+\frac{1}{4})\Gamma(s+\frac{7}{4})}Z^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s)+R^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s),\end{aligned}$$

*and when $n_{+\infty}<0$, we have*

$$\begin{aligned}&\left\langle \overline{\mathbf{U}_{0,(L)}(\cdot;\overline{s},-\tfrac{1}{2},1-m^{(L)},\overline{\mu_p},r)},\ \overline{\mathbf{U}_{(\ell)}(\cdot;s+2,-\tfrac{1}{2},1-n,\overline{\mu_p},r)}\right\rangle\\ &\quad=\frac{e(\frac{1}{8})\Gamma(2s+1)}{4^{s+1}\pi |n_{+\infty}|^2\Gamma(s-\frac{1}{4})\Gamma(s+\frac{9}{4})}Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)+R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s),\end{aligned}$$

$$\begin{aligned}&\left\langle \overline{\mathbf{U}_{0}(\cdot;\overline{s},-\tfrac{1}{2},\mathbf{1}-\mathbf{m},\overline{\mu_p},r)},\ \overline{\mathbf{U}_{(\ell)}(\cdot;s+2,-\tfrac{1}{2},1-n,\overline{\mu_p},r)}\right\rangle\\ &\quad=\frac{e(\frac{1}{8})\Gamma(2s+1)}{4^{s+1}\pi |n_{+\infty}|^2\Gamma(s-\frac{1}{4})\Gamma(s+\frac{9}{4})}Z^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s)+R^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s).\end{aligned}$$

*Here both $R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)$ and $R^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s)$ are holomorphic in $\sigma>\frac{1}{2}$ and*

$$R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)\ll_p \frac{|m_{+0}^{(L)}n_{+\infty}|}{\sigma-\frac{1}{2}},\qquad R^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s)\ll_p \frac{|Mn_{+\infty}|}{\sigma-\frac{1}{2}}.$$

*Proof.* Set $\operatorname{Re}s_2=\sigma_2>\operatorname{Re}s_1=\sigma_1>\frac{1}{2}$. We only prove the formulas for the inner products involving $\mathbf{U}_{0,(L)}(\cdot;\overline{s},-\frac{1}{2},1-m^{(L)},\overline{\mu_p},r)$, while the other two formulas for $\mathbf{U}_0(\cdot;\overline{s},-\frac{1}{2},\mathbf{1}-\mathbf{m},\overline{\mu_p},r)$ follow by summing on $L\in\rhd r\lhd$ and by $M=\max_{L\in\rhd r\lhd}|m_{+0}^{(L)}|$.

After we prove the formulas for these inner products, the meromorphic property of the Kloosterman-Selberg zeta functions follows directly from the meromorphic continuation (3.35) and (3.36).

We start with the first inner product using unfolding. Here $\overline{\mathbf{U}_{0,(L)}}$ abbreviates the first term in the inner product. Recall (3.29) and (3.16). We have

$$\begin{aligned}
&\left\langle \overline{\mathbf{U}_{0,(L)}(\cdot;\overline{s_1},-\tfrac{1}{2},m^{(L)},\overline{\mu_p},r)}, \mathbf{U}_{(\ell)}(\cdot;\overline{s_2},\tfrac{1}{2},n,\mu_p,r)\right\rangle \\
&= \int_{\Gamma_0(p)\backslash\mathbb{H}} \left( \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(p)} \mu_p(\gamma)^{-1} j(\gamma,z)^{-\frac{1}{2}} (\operatorname{Im}\gamma z)^{\overline{s_2}} \frac{e(n_{+\infty}\gamma z)}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell \right)^{\mathrm{H}} \overline{\mathbf{U}_{0,(L)}} \frac{dxdy}{y^2} \\
&= \sum_{\Gamma_\infty\backslash\Gamma_0(p)} \int_{\Gamma_0(p)\backslash\mathbb{H}} y^{s_2} e^{-2\pi n_{+\infty} y} \frac{e(-n_{+\infty}x)\mathfrak{e}_\ell^{\mathrm{T}}}{\sin(\frac{\pi\ell}{p})} \overline{\mathbf{U}_{0,(L)}(z;\overline{s_1},-\tfrac{1}{2},m^{(L)},\overline{\mu_p},r)} \frac{dxdy}{y^2}.
\end{aligned}$$

Then we apply the Fourier expansion of $\mathbf{U}_{0,(L)}(\cdot;\overline{s_1},-\frac{1}{2},m^{(L)},\overline{\mu_p},r)$ in (3.34) with (3.24) to continue:

$$\begin{aligned}
&= e(-\tfrac{1}{4}) \sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]=L}} \frac{\overline{S_{0\infty}^{(\ell)}(1-m^{(L)},1-n,a,\overline{\mu_p})}}{(a\sqrt{p})^{2s_1}} \int_0^\infty y^{s_2-s_1-1} e^{-2\pi n_{+\infty} y} \\
&\quad \cdot \int_{\mathbb{R}} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}} e\left(\frac{-m_{+0}^{(L)}(u+i)}{pa^2y(u^2+1)} - n_{+\infty} yu\right) \frac{du}{(u^2+1)^{s_1}} dy \\
&= \frac{e(-\frac{3}{8})4^{1-s_2}\pi^{1+s_1-s_2}}{n_{+\infty}^{s_2-s_1}} \cdot \frac{\Gamma(s_2+s_1-1)\Gamma(s_2-s_1)}{\Gamma(s_1+\frac{1}{4})\Gamma(s_2-\frac{1}{4})} Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1) + R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1,s_2).
\end{aligned}$$

Here

$$\begin{aligned}
R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1,s_2) &= e(-\tfrac{3}{8}) \sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]=L}} \frac{S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p)}{(a\sqrt{p})^{2s_1}} \int_0^\infty y^{s_2-s_1-1} e^{-2\pi n_{+\infty} y} \\
&\quad \cdot \int_{\mathbb{R}} \left(\frac{u+i}{|u+i|}\right)^{-\frac{1}{2}} e\left(-n_{+\infty}yu\right) \left( e\left(\frac{-m_{+0}^{(L)}(u+i)}{pa^2y(u^2+1)}\right) - 1\right) \frac{du}{(u^2+1)^{s_1}} dy \\
&= \sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]=L}} \frac{S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p)}{(a\sqrt{p})^{2s_1}} O_p\left(\frac{|m_{+0}^{(L)} n_{+\infty}|}{pa^2(\sigma_1-\frac{1}{2})}\right) = O_p\left(\frac{|m_{+0}^{(L)} n_{+\infty}|}{\sigma_1-\frac{1}{2}}\right)
\end{aligned}$$

and is holomorphic in the region $\operatorname{Re} s_1 > \frac{1}{2}$ with $s_2 = s_1 + 2$. The last step is by the trivial bound $S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p) \ll a$.

Then we deal with the case $n_{+\infty} < 0$. As before, we have

$$\begin{aligned}
&\left\langle \overline{\mathbf{U}_{0,(L)}(\cdot;\overline{s_1},-\tfrac{1}{2},1-m^{(L)},\overline{\mu_p},r)},\ \overline{\mathbf{U}_{(\ell)}(\cdot;s_2,-\tfrac{1}{2},1-n,\overline{\mu}_p,r)}\right\rangle \\
&= \int_{\Gamma_\infty\backslash\mathbb{H}} y^{s_2} \frac{e(-n_{+\infty}z)}{\sin(\frac{\pi\ell}{p})} \mathfrak{e}_\ell^{\mathrm{T}}\, \overline{\mathbf{U}_{0,(L)}(\cdot;\overline{s_1},-\tfrac{1}{2},1-m^{(L)},\overline{\mu_p},r)} \frac{dxdy}{y^2} \\
&= \frac{e(-\frac{3}{8})4^{1-s_2}\pi^{1+s_1-s_2}}{|n_{+\infty}|^{s_2-s_1}} \cdot \frac{\Gamma(s_2+s_1-1)\Gamma(s_2-s_1)}{\Gamma(s_1-\frac{1}{4})\Gamma(s_2+\frac{1}{4})} Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1) + R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1,s_2).
\end{aligned}$$

We still have that $R^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s_1,s_2) = O_p\left(\frac{|m_{+0}^{(L)} n_{+\infty}|}{\sigma_1-\frac{1}{2}}\right)$ and is holomorphic for $\sigma_1 > \frac{1}{2}$ and $s_2 = s_1 + 2$. This finishes the proof.

$\square$

The following proposition follows directly from Lemma 3.11, Cauchy-Schwarz, Lemma 3.10 and Lemma 3.7. Note that the norms involving $s+2$ and $\overline{s}+2$ have $\sigma+2-\frac{1}{2}>2$ and do not contribute to the denominator.

**Proposition 3.12.** *Let* $\mathbf{m}\leq 0$, $M=\max_{\ell\in\triangleright r\triangleleft}|m_{+0}^{(\ell)}|$ *and* $s=\sigma+it$ *with* $\operatorname{Re}s=\sigma>\frac{1}{2}$. *When* $|t|\to\infty$, *we have the growth condition*

$$Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)\ll_p\frac{|m_{+0}^{(L)}n_{+\infty}|^3|t|^{\frac{1}{2}}}{\sigma-\frac{1}{2}}\quad\textit{and}\quad Z^{(\ell)}_{\substack{0\infty;r\\ \mathbf{m},n,+}}(s)\ll_p\frac{|Mn_{+\infty}|^3|t|^{\frac{1}{2}}}{\sigma-\frac{1}{2}}.$$

Then we can prove the remaining bounds (3.12) and (3.13) in Theorem 3.5.

*Proof of* (3.12) *and* (3.13) *in Theorem 3.5.* Take any small $\varepsilon>0$. Since $Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(s)\ll_\varepsilon\zeta(1+\varepsilon)$ for $\operatorname{Re}s=1+\varepsilon$, by the Phragmén-Lindelöf principle, we have

$$Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}\left(\tfrac{1+s}{2}\right)\ll_{p,\varepsilon}|m^{(L)}n_{+\infty}|^3|t|^{\frac{1}{2}-\frac{\sigma}{2}+\varepsilon}\quad\text{for }0<\varepsilon\leq\sigma\leq 1+\varepsilon. \tag{3.41}$$

Following the similar step after (3.25), by the proof of prime number theorem as in [40, Chapter 18], we have

$$\sum_{\substack{a>0:\\ p\nmid a,\,[a\ell]=L}}\frac{S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p)}{a\sqrt{p}}=\frac{1}{2\pi i}\int_{1-iT}^{1+iT}Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}\left(\tfrac{s+1}{2}\right)\frac{x^s}{s}ds+O\left(\frac{|m_{+0}^{(L)}n_{+\infty}|^3x^{1+\varepsilon}}{T}\right). \tag{3.42}$$

By Lemma 3.11, the function $Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}(\frac{1+s}{2})$ has at most a finite number of simple poles at $2s_j-1\in(0,1)$ (note that we are using $\frac{1+s}{2}$ rather than $s$), where $\lambda_j=s_j(1-s_j)<\frac{1}{4}$ are the discrete eigenvalues of $\Delta_{\frac{1}{2}}$ on $\mathcal{L}_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$. Shifting the line of integration above to $\operatorname{Re}s=\varepsilon$ ($\varepsilon-iT\to\varepsilon+iT$) such that $2s_j-1>\varepsilon$ for all $\lambda_j<\frac{1}{4}$, with the help of (3.41) we obtain

$$\sum_{\substack{a>0:\\ p\nmid a,\,[a\ell]=L}}\frac{S_{0\infty}^{(\ell)}(m^{(L)},n,a,\mu_p)}{a\sqrt{p}}=\sum_{\frac{1}{2}<s_j<\frac{3}{4}}\operatorname*{Res}_{s=2s_j-1}Z^{(\ell)}_{\substack{0\infty;r\\ m^{(L)},n,+}}\left(\tfrac{1+s}{2}\right)\frac{x^{2s_j-1}}{2s_j-1}+O(|m_{+0}^{(L)}n_{+\infty}|^3x^{\frac{1}{3}+\varepsilon}),$$

where we have chosen $T=x^{\frac{2}{3}}$.

For the residues, by Lemma 3.11, it suffices to compute the residue of the first and third inner products in that lemma. Let $\mathrm{OB}(r_j)$ be an orthonormal basis of $\tilde{\mathcal{L}}_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$ and let $\mathbf{V}(z;r_j)\in\mathrm{OB}(r_j)$. Combining Lemma 3.11, (3.39), (3.26), and (3.27), the proof of (3.12) follows.

The proof of (3.13) follows by summing up on $L\in\triangleright r\triangleleft$ and by $M=\max_{L\in\triangleright r\triangleleft}|m_{+0}^{(L)}|$.

$\square$

### 3.2.3 Convergence

In this subsection we show the growth rates and convergence properties of sums of Kloosterman sums. We will need these estimates in Chapter 7.

**Proposition 3.13.** *With the same setting as Theorem 3.5, we have that all the following sums are convergent with bound*

$$\sum_{\substack{c>4\pi|m_{+\infty}n_{+\infty}|^{\frac12}\\ p|c}} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}\mathscr{M}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac12}}{c}\right) \ll_p |m_{+\infty}n_{+\infty}|^4,$$

$$\sum_{\substack{a>4\pi\left|m^{(L)}_{+0}n_{+\infty}\right|^{\frac12}\\ p\nmid a,\,[a\ell]=L}} \left|\frac{m^{(L)}_{+0}}{n_{+\infty}}\right|^{\frac14} \frac{S^{(\ell)}_{0\infty}(m^{(L)},n,a,\mu_p)}{a\sqrt{p}}\mathscr{M}\left(\frac{4\pi|m^{(L)}_{+0}n_{+\infty}|^{\frac12}}{c}\right) \ll_p |m^{(L)}_{+0}n_{+\infty}|^4,$$

$$\sum_{\substack{a>4\pi|Mn_{+\infty}|^{\frac12}\\ p\nmid a,}} \sum_{\ell\in\rhd a,r\lhd} \left|\frac{m^{([a\ell])}_{+0}}{n_{+\infty}}\right|^{\frac14} \frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p)}{a\sqrt{p}}\mathscr{M}\left(\frac{4\pi|m^{([a\ell])}_{+0}n_{+\infty}|^{\frac12}}{c}\right) \ll_p |Mn_{+\infty}|^4.$$

*Here $\mathscr{M}$ is either the Bessel function $I_{\frac12}$ or $J_{\frac12}$.*

*Proof.* The proof is similar to the scalar-valued case. We use the properties of Bessel functions: by [31, (10.14.4)], $|J_\alpha(x)| \le_\alpha x^\alpha$ for $x>0$ and $\alpha \ge -\frac12$, and by [31, (10.30.1)], for $0\le x\le \beta$,

$$I_\alpha(x) \ll_{\alpha,\beta} x^\alpha \quad \text{for } \alpha > -1.$$

Let $\phi := 4\pi|m_{+\infty}n_{+\infty}|^{\frac12}$. We have $\phi \ge \frac{\pi}{6}$ by $\alpha_{+\infty} = \frac{1}{24}$ in (2.35). When $t\ge\phi$, we have $0\le \frac{\phi}{t}\le 1$, hence by [31, (10.29.1), (10.6.1)], we get

$$\frac{d}{dt}I_{\frac12}\left(\frac{\phi}{t}\right) = -\frac{\phi}{2t^2}\left(I_{-\frac12}\left(\frac{\phi}{t}\right)+I_{\frac32}\left(\frac{\phi}{t}\right)\right) \ll \phi^{\frac12}t^{-\frac32}+\phi^{\frac52}t^{-\frac72} \ll \phi^{\frac12}t^{-\frac32}, \tag{3.43}$$

$$\frac{d}{dt}J_{\frac12}\left(\frac{\phi}{t}\right) = -\frac{\phi}{2t^2}\left(J_{-\frac12}\left(\frac{\phi}{t}\right)-J_{\frac32}\left(\frac{\phi}{t}\right)\right) \ll \phi^{\frac12}t^{-\frac32}+\phi^{\frac52}t^{-\frac72} \ll \phi^{\frac12}t^{-\frac32}. \tag{3.44}$$

By Corollary 3.6, we write

$$\mathrm{SC}(x) := \sum_{p|c\le x} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} \ll_p |m_{+\infty}n_{+\infty}|^3 x^{\frac12-\delta}.$$

By partial integration, for $T>\phi$ we have

$$\begin{aligned}\sum_{\substack{\phi<c\le T\\ p|c}} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}\mathscr{M}\left(\frac{\phi}{c}\right) &\ll \left|\mathscr{M}\left(\frac{\phi}{T}\right)\mathrm{SC}(T)\right| + |\mathscr{M}(1)\mathrm{SC}(\phi)| + \left|\int_\phi^T \mathrm{SC}(t)\phi^{\frac12}t^{-\frac32}\,dt\right| \\ &\ll |m_{+\infty}n_{+\infty}|^3\phi^{\frac12}(1+T^{-\delta}) \\ &\ll |m_{+\infty}n_{+\infty}|^4.\end{aligned} \tag{3.45}$$

For $Y > X > \phi$, we also have

$$\sum_{\substack{X<c\le Y\\ p|c}} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}\mathscr{M}\left(\frac{\phi}{c}\right) \ll \left|\mathscr{M}\left(\frac{\phi}{Y}\right)\mathrm{SC}(Y)-\mathscr{M}\left(\frac{\phi}{X}\right)\mathrm{SC}(X)\right| + \left|\int_X^Y \mathrm{SC}(t)\phi^{\frac12}t^{-\frac32}dt\right| \\ \ll |m_{+\infty}n_{+\infty}|^3\phi^{\frac12}\left|Y^{-\delta}-X^{-\delta}\right|. \tag{3.46}$$

The above estimate (3.46) proves the convergence by Cauchy's criterion, so we are able to let $T\to\infty$ in (3.45). We have proved the first bound of the proposition.

Again by Corollary 3.6, we write

$$\mathrm{SA}(x) := \sum_{\substack{a\le x:\\ p\nmid a,\ [a\ell]=L}} \frac{S^{(\ell)}_{0\infty}(m^{(L)},n,a,\mu_p)}{a\sqrt{p}} \ll_p |m^{(L)}_{+0}n_{+\infty}|^3 x^{\frac12-\delta}.$$

Let $\phi := 4\pi|m^{(L)}_{+0}n_{+\infty}|^{\frac12}$. A similar application of the partial sum concludes the second formula. The third formula follows from summing up $L\in\triangleright r\triangleleft$ and by $M=\max_{L\in\triangleright r\triangleleft}|m^{(L)}_{+0}|$. Here we can re-order the sum because the convergence can be easily derived by separating the partial sum into $\#\triangleright r\triangleleft$ parts. □

For $\beta>0$, let $\Gamma(\alpha;\beta):=\int_\beta^\infty t^{\alpha-1}e^{-t}dt$ be the incomplete Gamma function. We have $\Gamma(\alpha;x)\sim x^{\alpha-1}e^{-x}$ when $x\to\infty$.

**Proposition 3.14.** *With the same setting as Proposition 3.13, we have*

$$\sum_{c>0:\,p|c}\frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}I_{\frac12}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac12}}{c}\right) \ll_p |m_{+\infty}n_{+\infty}|^5 e^{4\pi|m_{+\infty}n_{+\infty}|^{\frac12}},$$

$$\sum_{c>0:\,p|c}\frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}J_{\frac12}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac12}}{c}\right) \ll_p |m_{+\infty}n_{+\infty}|^5,$$

$$\sum_{\substack{a>0\\ p\nmid a}}\sum_{\ell\in\triangleright a,r\triangleleft}\left|\frac{m^{([a\ell])}_{+0}}{n_{+\infty}}\right|^{\frac14}\frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p)}{a\sqrt{p}}I_{\frac12}\left(\frac{4\pi|m^{([a\ell])}_{+0}n_{+\infty}|^{\frac12}}{a\sqrt{p}}\right) \ll_p |Mn_{+\infty}|^5 e^{4\pi|Mn_{+\infty}|^{\frac12}},$$

$$\sum_{\substack{a>0\\ p\nmid a}}\sum_{\ell\in\triangleright a,r\triangleleft}\left|\frac{m^{([a\ell])}_{+0}}{n_{+\infty}}\right|^{\frac14}\frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p)}{a\sqrt{p}}J_{\frac12}\left(\frac{4\pi|m^{([a\ell])}_{+0}n_{+\infty}|^{\frac12}}{a\sqrt{p}}\right) \ll_p |Mn_{+\infty}|^5.$$

*Hence, by denoting $q = e(z)$ with $y = \operatorname{Im} z > 0$, the following series converge:*

$$\sum_{n_{+\infty}>0} \left|\frac{m_{+\infty}}{n_{+\infty}}\right|^{\frac{1}{4}} \left| \sum_{c>0:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} I_{\frac{1}{2}}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac{1}{2}}}{c}\right)\right| |q^{n_{+\infty}}|,$$

$$\sum_{n_{+\infty}<0} \left|\frac{m_{+\infty}}{n_{+\infty}}\right|^{\frac{1}{4}} \left| \sum_{c>0:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} J_{\frac{1}{2}}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac{1}{2}}}{c}\right)\right| \left|\Gamma(\tfrac{1}{2}, 4\pi|n_{+\infty}|y) q^{n_{+\infty}}\right|,$$

$$\sum_{n_{+\infty}>0} \left| \sum_{\substack{a>0:\, p\nmid a,\\ \ell\in\rhd a, r\lhd}} \sum \left|\frac{m^{([a\ell])}_{+0}}{n_{+\infty}}\right|^{\frac{1}{4}} \frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p)}{a\sqrt{p}} I_{\frac{1}{2}}\left(\frac{4\pi|m^{([a\ell])}_{+0}n_{+\infty}|^{\frac{1}{2}}}{c}\right)\right| |q^{n_{+\infty}}|,$$

$$\sum_{n_{+\infty}<0} \left| \sum_{\substack{a>0:\, p\nmid a,\\ \ell\in\rhd a, r\lhd}} \sum \left|\frac{m^{([a\ell])}_{+0}}{n_{+\infty}}\right|^{\frac{1}{4}} \frac{S^{(\ell)}_{0\infty}(m^{([a\ell])},n,a,\mu_p)}{a\sqrt{p}} J_{\frac{1}{2}}\left(\frac{4\pi|m^{([a\ell])}_{+0}n_{+\infty}|^{\frac{1}{2}}}{c}\right)\right| \left|\Gamma(\tfrac{1}{2}, 4\pi|n_{+\infty}|y) q^{n_{+\infty}}\right|.$$

*Proof.* We first prove the formulas involving $I_{\frac{1}{2}}$. We refer to [31, (10.30.4)] that $I_\beta(\frac{\phi}{t}) \ll_\beta e^{\phi}(t/\phi)^{\frac{1}{2}}$ when $t \le \phi$. For the first formula, we let $\phi = 4\pi|m_{+\infty}n_{+\infty}|^{\frac{1}{2}}$ and use partial summation with [31, (10.29.1)] to get

$$\sum_{1\le c\le\phi:\, p|c} \frac{S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c} I_{\frac{1}{2}}\left(\frac{4\pi|m_{+\infty}n_{+\infty}|^{\frac{1}{2}}}{c}\right) \ll |m_{+\infty}n_{+\infty}|^3 e^{\phi}\phi^{\frac{3}{2}-\delta} \ll |m_{+\infty}n_{+\infty}|^5 e^{\phi}.$$

We combine this with Proposition 3.13 to get the desired bound. Since $|e(n_{+\infty}z)| = e^{-2\pi n_{+\infty}y}$, and considering that the inner sum grows at a rate of $e^{O(n_{+\infty}^{1/2})}$, the fifth formula is clear.

To prove the third formula, we start by choosing a fixed $L \in\rhd r \lhd$. Then we set $\phi = 4\pi|m^{(L)}_{+0}n_{+\infty}|^{\frac{1}{2}}$ and apply the second formula from Corollary 3.6. Adding for all $L \in\rhd r \lhd$, we get the desired bounds here. The seventh formula follows from the same reason as the fifth one.

For the formulas involving $J_{\frac{1}{2}}$ we apply (3.44) and get the similar conclusion without the exponential growth term $e^{\phi}$. Since $n_{+\infty} < 0$, we have

$$\left|\Gamma(\tfrac{1}{2}, 4\pi|n_{+\infty}|y) e(n_{+\infty}z)\right| \ll e^{-2\pi|n_{+\infty}|y}$$

and still get the convergence as $y > 0$. □

*Remark.* The step of repeating the selection of $L$ and then summing up on $L \in\rhd r \lhd$ is necessary. This is because $m^{([a\ell])}_{+0}$ changes when $a$ varies, but it remains constant when we specify $[a\ell] = L$.

# Chapter 4

# Sums of Kloosterman sums: uniform bounds, mixed-sign case

In Chapter 4 and Chapter 5 we will prove Theorem 1.7. Since the trace formula has different settings in the mixed-sign case $\tilde{m}\tilde{n} < 0$ and the same-sign case $\tilde{m}\tilde{n} > 0$, we separate the two chapters. The proof in this chapter is contained in [14].

Before we prove Theorem 1.7, we need a few preparations.

## 4.1 Examples of admissible multipliers and a lower bound of the spectrum

Suppose $N$ is a positive integer. In this section we are going to prove the following proposition and conclude a lower bound for the exceptional spectrum in certain cases.

**Proposition 4.1.** *If $\nu = \chi\nu_\theta$ or $\nu = \chi\nu_\eta$ where $\chi$ is a real character modulo $N$, then $\nu$ and its conjugate are admissible, i.e. satisfy the requirements in Definition 1.6. If $\nu$ is the multiplier for a weight $\pm\frac{1}{2}$ eta-quotient, then $\nu$ satisfies the condition (1) in Definition 1.6.*

We will verify this proposition in the next subsection while we only prove the case for weight $\frac{1}{2}$. The proof for weight $-\frac{1}{2}$ case with respect to the conjugate of each multiplier follows from the same process. For simplicity we recall Dirichlet's lemma:

**Lemma 4.2.** *Every real character $\chi$ modulo $N$ can be expressed in the form $\chi(y) = (\frac{d}{y})$ where $d \equiv 0, 1 \pmod 4$ depends on $\chi$ and $N$. Every primitive real character can be expressed in the form*

$$\chi(y) = (\tfrac{D}{y}),$$

*where $D$ is a fundamental discriminant and $|D|$ equals the conductor of $\chi$.*

### 4.1.1 Proof of Proposition 4.1

Suppose $\chi$ is a quadratic character modulo $N$. If $\nu = \chi\nu_\theta$, write $\chi = (\frac{D}{\cdot})\mathbf{1}_{N/D}$ where $D$ is a fundamental discriminant. Since $\nu$ is assumed to be a weight $\frac{1}{2}$ multiplier, we have $\chi(-1) = 1$ so $D > 0$. If $D \equiv 0 \pmod 4$, we are done; if $D \equiv 1 \pmod 4$, then $-4D$ is fundamental and $\nu$ equals $(\frac{|-4D|}{\cdot})\nu_\theta$ on $\Gamma_0(N)$. Now we have proved condition (1) for $\nu = \chi\nu_\theta$.

The individual Weil bound is known by Blomer [42, (2.15)] that for $4|N|c$, we have

$$|S(m,n,c,\nu)| \le \sigma_0(c)(m,n,c)^{\frac12} c^{\frac12},$$

where $\nu = \psi\nu_\theta$ or $\overline{\psi\nu_\theta}$ for a Dirichlet character $\psi$ modulo $N$ satisfying $\psi(-1) = 1$. This proves condition (2) for $\nu = \chi\nu_\theta$.

For $\nu = \chi\nu_\eta$, we have a map to $\tilde{\mathcal{L}}_{\frac12}\left(576N, (\frac{12}{\cdot})\chi\nu_\theta\right)$:

**Lemma 4.3.** *For $\nu = \chi\nu_\eta$ and for each $r$, the map $z \to 24z$ gives an injection*

$$\tilde{\mathcal{L}}_{\frac12}(N,\nu,r) \to \tilde{\mathcal{L}}_{\frac12}\left(576N, (\tfrac{12}{\cdot})\chi\nu_\theta, r\right).$$

*Proof.* The proof follows from a similar process as [12, Lemma 3.2] by setting $\gamma' = \left(\begin{smallmatrix} a & 24b \\ c/24 & d\end{smallmatrix}\right)$ when $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(576N)$, $c > 0$. For any $f \in \tilde{\mathcal{L}}_{\frac12}(N,\nu,r)$, define

$$g(z) := f(24z) = f|_{\frac12}\left(\begin{pmatrix} \sqrt{24} & 0 \\ 0 & 1/\sqrt{24}\end{pmatrix}\right).$$

Observe that

$$g|_{\frac12}\gamma = f|_{\frac12}\gamma'\begin{pmatrix} \sqrt{24} & 0 \\ 0 & 1/\sqrt{24}\end{pmatrix}.$$

One can check that $(\chi\nu_\eta)(\gamma') = \chi(d)(\frac{12}{d})\nu_\theta(\gamma)$ by (1.19) with the help of the identities $e(\frac{1-d}{8}) = (\frac{2}{d})\varepsilon_d$ and $\varepsilon_d^2 = (\frac{-1}{d})$ for odd $d$. ☐

Now $(\frac{12}{\cdot})\chi\nu_\theta$ is a weight $\frac12$ multiplier on $\Gamma_0(M) = \Gamma_0(576N)$ and $\chi(-1) = 1$. As in the beginning of this subsection, $(\frac{12}{\cdot})\chi\nu_\theta$ equals $(\frac{D'}{\cdot})\nu_\theta$ on $\Gamma_0(M)$ for some $D'$ fundamental. Finally we pick $(\frac{D'}{\cdot})\nu_\theta$ or $(\frac{|-4D'|}{\cdot})\nu_\theta$.

Although we only need an average bound, we have an individual Weil bound for $\nu = \chi\nu_\eta$.

**Lemma 4.4.** *Suppose that $\nu = \psi_q\nu_\eta$ where $\psi_q$ is a Dirichlet character modulo $q$ with $q|N|c$. Write $24m - 23 = \alpha^2 M_1$ with $M_1$ square-free. Then we have*

$$|S(m,n,c,\nu)| \ll q^{\frac32}\sigma_0((\alpha,c))\sigma_0(c)\sqrt{c}\cdot((24m-23)(24n-23),c)^{\frac12}.$$

*Proof.* Set $r = N/q$ and $s = c/N$. By (1.11) we have $\alpha_\nu = \alpha_{\nu_\eta}$ so $\tilde n = n - \frac{23}{24}$. We have

$$\begin{aligned} S(m,n,c,\nu) &= \sum_{d \pmod c)^*} \overline{\psi}_q(d)\overline{\nu_\eta}(\gamma) e\left(\frac{\tilde m a + \tilde n d}{c}\right) \\ &= \sum_{d \pmod c)^*} \left(\sum_{\ell=1}^q \frac{\overline{\psi}_q(\ell)}{q}\sum_{h=1}^q e\left(\frac{h(d-\ell)}{q}\right)\right)\overline{\nu_\eta}(\gamma)e\left(\frac{\tilde m a + \tilde n d}{c}\right) \\ &= \frac1q \sum_{\ell=1}^q \overline{\psi}_q(\ell)\sum_{h=1}^q e\left(-\frac{h\ell}{q}\right) \sum_{d \pmod c)^*} \overline{\nu_\eta}(\gamma) e\left(\frac{\tilde m a + (\tilde n + hrs)d}{c}\right). \end{aligned}$$

The proof now follows from the Weil-type bound for $S(m,n,c,\nu_\eta)$. By [11, Proposition 2.1] we see that

$$|S(m,n,c,\nu)| \ll q\sigma_0((\alpha,c))\sigma_0(c)\sqrt{c}\cdot \max_{1\le h\le q}(M_1N_2,c)^{\frac12},$$

where $N_1 = 24n - 23$ and $N_2 = 24(n + hrs) - 23 = 24n - 23 + \frac{24hc}{q}$. We finish the proof by a rough estimate

$$(M_1 N_2, c) \le q(M_1 N_2, \tfrac{c}{q}) = q(M_1 N_1, \tfrac{c}{q}) \le q(M_1 N_1, c).$$

□

It remains to prove the claim for eta-quotients in Proposition 4.1.

**Lemma 4.5.** *If $\nu$ is the multiplier system for an eta-quotient*

$$f(z) = \prod_{\delta | L} \eta(\delta z)^{r_\delta}$$

*of weight $\frac{1}{2} = \frac{1}{2}\sum_{\delta|L} r_\delta$, then the map $z \to 24z$ gives an injection*

$$\tilde{\mathcal{L}}_{\frac{1}{2}}(N, \nu, r) \to \tilde{\mathcal{L}}_{\frac{1}{2}}\left(576LN,\ \prod_{\delta|L}\left(\frac{12\delta}{\cdot}\right)^{r_\delta}\nu_\theta,\ r\right).$$

*Proof.* The proof is similar to Lemma 4.3. Let $\nu_\delta$ denote the multiplier for a factor $\eta(\delta z)$. Since for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(576\delta)$,

$$\left(\begin{smallmatrix} 24\delta & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & 24\delta b \\ c/24\delta & d \end{smallmatrix}\right)\left(\begin{smallmatrix} 24\delta & 0 \\ 0 & 1 \end{smallmatrix}\right),$$

we have

$$\nu_\delta\left(\left(\begin{smallmatrix} a & 24b \\ c/24 & d \end{smallmatrix}\right)\right) = \nu_\eta\left(\left(\begin{smallmatrix} a & 24\delta b \\ c/24\delta & d \end{smallmatrix}\right)\right) = \left(\tfrac{c/24\delta}{d}\right) e\left(\tfrac{d-1}{8}\right) = \left(\tfrac{\delta}{d}\right)\left(\tfrac{12}{d}\right)\nu_\theta\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)$$

because $e(\frac{1-d}{8}) = (\frac{2}{d})\varepsilon_d$ when $d$ is odd. We take the product of all the factors. □

*Remark.* For the multiplier of a eta-quotient, the author does not know its Weil bound in general.

### 4.1.2 Lower bound for the exceptional spectrum

After we get a twisted theta-multiplier by level lifting, the following theorems show the relationship of eigenvalues between weight 0 and weight $\frac{1}{2}$ eigenforms.

**Theorem 4.6** ([43, p. 304])**.** *Let $\chi$ be a Dirichlet character modulo $4N'$ for a positive integer $N'$, $\nu = \nu_\theta\left(\frac{N'}{\cdot}\right)\overline{\chi}$, then for each eigenvalue $\lambda = \frac{1}{4} + r^2$ of $\Delta_{\frac{1}{2}}$ for $(\Gamma_0(4N'), \nu)$, there is an eigenvalue $\lambda' = \frac{1}{4} + 4r^2$ of $\Delta_0$ for $(\Gamma_0(2N'), \chi^2)$.*

Recall the definition (2.16) of $r_\Delta(N, \nu, k)$. We have the following bound:

**Proposition 4.7.** *Let $\nu$ be a weight $k = \pm\frac{1}{2}$ multipler of $\Gamma = \Gamma_0(N)$ satisfying condition (1) in Definition 1.6 and assume $H_\theta$ (2.15). Then we have*

$$2\,\mathrm{Im}\, r_\Delta(N, \nu, k) \le \theta.$$

*Proof.* We prove the case for $k = \frac{1}{2}$ and the other case follows by conjugation. Condition (1) gives the injection

$$z \to Bz: \quad \tilde{\mathcal{L}}_{\frac{1}{2}}(N, \nu, r) \to \tilde{\mathcal{L}}_{\frac{1}{2}}\left(M, \left(\tfrac{|D|}{\cdot}\right)\nu_\theta, r\right)$$

where $4|N|M$. We set $\chi = (\frac{M}{\cdot})(\frac{|D|}{\cdot})$ and apply Theorem 4.6 to get an eigenvalue $\lambda' = \frac{1}{4} + r'^2$ of $\Delta_0$ for $(\Gamma_0(\frac{M}{2}), \mathbf{1})$ with eigenparameter

$$r' = r_\Delta(-\tfrac{M}{2}, \mathbf{1}, 0) = 2r_\Delta(N, \nu, \tfrac{1}{2}).$$

Assuming $H_\theta$ (2.15) we have $\operatorname{Im} r' \le \theta$ and finish the proof. □

## 4.2 Kuznetsov trace formula in the mixed-sign case

Let $\mathfrak{a}$ be a singular cusp for the weight $k$ multiplier system $\nu$ on $\Gamma = \Gamma_0(N)$. For $\operatorname{Re} s > 1$, define the Eisenstein series associated to $\mathfrak{a}$ as in [33], [35] by

$$E_\mathfrak{a}(z,s) := \sum_{\gamma\in\Gamma_\mathfrak{a}\backslash\Gamma} \overline{\nu(\gamma)w(\sigma_\mathfrak{a}^{-1},\gamma)}(\operatorname{Im}\sigma_\mathfrak{a}^{-1}\gamma z)^s j(\sigma_\mathfrak{a}^{-1}\gamma, z)^{-k} \tag{4.1}$$

and the Poincaré series for $m > 0$ by

$$\mathcal{U}_m(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\nu}(\gamma)(\operatorname{Im}\gamma z)^s j(\gamma,z)^{-k} e(\tilde{m}\gamma z). \tag{4.2}$$

Both of the series converge absolutely and uniformly on compact subsets of the fundamental domain $\Gamma\setminus\mathbb{H}$ when $\operatorname{Re} s > 1$ and both of them are automorphic functions of weight $k$ as functions of $z$. The Eisenstein series can be meromorphically extended to the entire $s$-plane and have Fourier expansions on $s = \frac{1}{2} + ir$ for $r \in \mathbb{R}$ ([33, (12-14)])

$$\begin{aligned} E_\mathfrak{a}(x+iy,s) &= \delta_{\mathfrak{a}\infty}y^s + \rho_\mathfrak{a}(0,r)y^{1-s} + \sum_{\ell\neq 0}\rho_\mathfrak{a}(\ell,r)W_{\frac{k}{2}\operatorname{sgn}\tilde{\ell},\,-ir}(4\pi|\tilde{\ell}|y)e(\tilde{\ell}x) \\ &= \delta_{\mathfrak{a}\infty}y^s + \frac{\delta_{\alpha_\nu 0}\cdot 4^{1-s}\Gamma(2s-1)}{e^{\frac{\pi ik}{2}}\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})}y^{1-s}\varphi_{\mathfrak{a}0}(s) \\ &\quad + \sum_{\ell\neq 0}|\tilde{\ell}|^{s-1}\frac{\pi^s W_{\frac{k}{2}\operatorname{sgn}\tilde{\ell},\,-ir}(4\pi|\tilde{\ell}|y)}{e^{\frac{\pi ik}{2}}\Gamma(s+\frac{k}{2}\operatorname{sgn}\tilde{\ell})}\varphi_{\mathfrak{a}\ell}(s)e(\tilde{\ell}x) \end{aligned} \tag{4.3}$$

where

$$\varphi_{\mathfrak{a}\ell}(s) = \sum_{c>0}\frac{1}{c^{2s}}\sum_{\substack{0\le d<c\\ \gamma=\left(\begin{smallmatrix}* & *\\ c & d\end{smallmatrix}\right)\in\sigma_\mathfrak{a}^{-1}\Gamma}} \overline{\nu}(\sigma_\mathfrak{a}\gamma)\overline{w_k}(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma)e\Big(\frac{\tilde{\ell}d}{c}\Big), \quad \ell\neq 0.$$

We introduce two different notations $\rho_\mathfrak{a}(\ell,r)$ and $\varphi_{\mathfrak{a}\ell}(s)$ for later convenience. The Fourier expansion of $E_\mathfrak{a}(z,s)$ at the cusp $\mathfrak{b}$ is denoted as [38, (2.64)] [44, p. 1551]

$$(E_\mathfrak{a}(\cdot,s)|_k\sigma_\mathfrak{b})(z) = \delta_{\mathfrak{a}\mathfrak{b}}y^s + \rho_{\mathfrak{a}\mathfrak{b}}(0,s)y^{1-s} + \sum_{\ell\neq 0}\rho_{\mathfrak{a}\mathfrak{b}}(\ell,s)W_{\frac{k}{2}\operatorname{sgn}\ell_\mathfrak{b},\,\frac{1}{2}-s}(4\pi|\ell_\mathfrak{b}|y)e(\ell_\mathfrak{b}x). \tag{4.4}$$

where $\rho_{\mathfrak{a}\mathfrak{b}}(0,s) = 0$ when $n_\mathfrak{b} \neq 0$. The Fourier expansion of the Poincaré series can be given by [11, (4.5)]

$$\mathcal{U}_m(x+iy,s) = y^s e(\tilde{m}z) + y^s\sum_{\ell\in\mathbb{Z}}\sum_{c>0}\frac{S(m,\ell,c,\nu)}{c^{2s}}B(c,\tilde{m},\tilde{\ell},y,s,k)e(\tilde{\ell}x). \tag{4.5}$$

where

$$B(c,\tilde{m},\tilde{\ell},y,s,k)=y\int_{-\infty}^{\infty}e\left(\frac{-\tilde{m}}{c^2y(u+i)}\right)\left(\frac{u+i}{|u+i|}\right)^{-k}\frac{e(-\tilde{\ell}yu)}{y^{2s}(u^2+1)^s}du.$$

When $\operatorname{Re} s>1$, we have $\mathcal{U}_m(\cdot,s)\in\mathcal{L}_k(N,\nu)$. More properties of these two series can be found in [33].

The following notations are very important in the remaining part of this chapter:

**Setting 4.8.** *Let $a=4\pi\sqrt{|\tilde{m}\tilde{n}|}\neq 0$ and $0<T\leq\frac{x}{3}$ with $T\asymp x^{1-\delta}$ where $\delta\in(0,\frac{1}{2})$ and finally will be chosen as $\frac{1}{3}$.*

We define a family of test functions $\phi:=\phi_{a,x,T}$ as in [19] and [11]:

**Setting 4.9.** *The test function $\phi:[0,\infty)\to\mathbb{R}$ is four times continuously differentiable and satisfies*

1. *$\phi(0)=\phi'(0)=0$, and for some $\varepsilon>0$,*

$$\phi^{(j)}(x)\ll_\varepsilon x^{-2-\varepsilon}\quad (j=0,\cdots,4)\quad \text{as } x\to\infty.$$

2. *$\phi(t)=1$ for $\frac{a}{2x}\leq t\leq\frac{a}{x}$.*
3. *$\phi(t)=0$ for $t\leq\frac{a}{2x+2T}$ and $t\geq\frac{a}{x-T}$.*
4. *$\phi'(t)\ll\left(\frac{a}{x-T}-\frac{a}{x}\right)^{-1}\ll\frac{x^2}{aT}$.*
5. *$\phi$ and $\phi'$ are piecewise monotone on a fixed number of intevals.*

Using the notation

$$\xi_{\mathfrak{a}}(r,f):=\int_{\Gamma\backslash\mathbb{H}}f(z)\overline{E_{\mathfrak{a}}\left(z,\tfrac{1}{2}+ir\right)}\frac{dxdy}{y^2},$$

we have the spectral theorem:

**Theorem 4.10** ([35, Theorem 2.1])**.** *Let $\{v_j(z)\}$ be an orthonormal basis of $\tilde{\mathcal{L}}_k(N,\nu)$. Then, any $f\in\mathcal{B}_k(\Gamma,\nu)$ has the expansion*

$$f(z)=\sum_j\langle f,v_j\rangle v_j(z)+\sum_{\text{singular }\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\xi_{\mathfrak{a}}(r,f)E_{\mathfrak{a}}\left(z,\tfrac{1}{2}+ir\right)dr$$

*which converges absolutely.*

We also have Parseval's identity [33, (27)]: for $f_1,f_2\in\mathcal{L}_k(\Gamma,\nu)$,

$$\langle f_1,f_2\rangle=\sum_{r_j}\langle f_1,v_j\rangle\overline{\langle f_2,v_j\rangle}+\sum_{\text{singular }\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\xi_{\mathfrak{a}}(r,f)\overline{\xi_{\mathfrak{a}}(r,f_2)}dr. \tag{4.6}$$

Define

$$\check{\phi}(r):=\operatorname{ch}\pi r\int_0^\infty K_{2ir}(u)\phi(u)\frac{du}{u} \tag{4.7}$$

which is an even function for $r\in\mathbb{R}$. Here we prove a Kuznetsov trace formula in the mixed-sign case:

**Theorem 4.11.** *Suppose $\nu$ is a multiplier system of weight $k=\pm\frac{1}{2}$ on $\Gamma=\Gamma_0(N)$. Let $\{v_j(\cdot)\}$ be an orthonormal basis of $\tilde{\mathcal{L}}_k(N,\nu)$. Let $\rho_j(n)$ denote the $n$-th Fourier coefficient of $v_j(\cdot)$. For each singular cusp*

$\mathfrak{a}$ *of* $(\Gamma,\nu)$*, let* $E_{\mathfrak{a}}(\cdot,s)$ *be the associated Eisenstein series. Let* $\varphi_{\mathfrak{a}n}(\frac{1}{2}+ir)$ *and* $\rho_{\mathfrak{a}}(n,r)$ *be defined as in* (4.3)*. Then for* $\tilde{m}>0$ *and* $\tilde{n}<0$ *we have*

$$\frac{i^{-k}}{2}\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}\phi\left(\frac{4\pi\sqrt{\tilde{m}|\tilde{n}|}}{c}\right)=4\sqrt{\tilde{m}|\tilde{n}|}\sum_{r_j}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)+\sum_{\text{singular }\mathfrak{a}}\mathcal{E}_{\mathfrak{a}},\tag{4.8}$$

*where*

$$\begin{aligned}\mathcal{E}_{\mathfrak{a}}&=\int_{-\infty}^{\infty}\left(\frac{\tilde{m}}{|\tilde{n}|}\right)^{-ir}\frac{\overline{\varphi_{\mathfrak{a}m}\left(\frac{1}{2}+ir\right)}\varphi_{\mathfrak{a}n}\left(\frac{1}{2}+ir\right)\check{\phi}(r)}{\Gamma\left(\frac{1}{2}+\frac{k}{2}-ir\right)\Gamma\left(\frac{1}{2}-\frac{k}{2}+ir\right)\operatorname{ch}\pi r}dr\\&=4\sqrt{\tilde{m}|\tilde{n}|}\cdot\frac{1}{4\pi}\int_{-\infty}^{\infty}\overline{\rho_{\mathfrak{a}}\left(m,r\right)}\rho_{\mathfrak{a}}\left(n,r\right)\frac{\check{\phi}(r)}{\operatorname{ch}\pi r}dr.\end{aligned}$$

*Remark.* The last equality follows from the following identity

$$\sqrt{\frac{|\tilde{n}|}{\pi}}\rho_{\mathfrak{a}}(n,r)=\frac{e^{-\frac{\pi ik}{2}}\pi^{ir}|\tilde{n}|^{ir}}{\Gamma\left(\frac{1}{2}+ir+\frac{k}{2}\operatorname{sgn}\tilde{n}\right)}\varphi_{\mathfrak{a}n}\left(\frac{1}{2}+ir\right).\tag{4.9}$$

*Proof.* The proof follows the outline of [11, Section 4], taking into account the contribution of the continuous spectrum. When $n\neq\tilde{n}$, i.e. $\alpha_\nu>0$, as in [11, Lemma 4.2, Lemma 4.3], for $\operatorname{Re}s_1>1$ and $\operatorname{Re}s_2>1$ we have

$$\begin{aligned}I_{m,n}(s_1,s_2)&:=\left\langle\mathcal{U}_m\left(\cdot,s_1,k,\nu\right),\overline{\mathcal{U}_{1-n}\left(\cdot,s_2,-k,\overline{\nu}\right)}\right\rangle\\&=2^{3-s_1-s_2}\left(\frac{\tilde{m}}{|\tilde{n}|}\right)^{\frac{s_2-s_1}{2}}\frac{i^{-k}\pi\Gamma(s_1+s_2-1)}{\Gamma\left(s_1-\frac{k}{2}\right)\Gamma\left(s_2+\frac{k}{2}\right)}\sum_{c>0}\frac{S(m,n,c,\nu)}{c^{s_1+s_2}}K_{s_1-s_2}\left(\frac{4\pi\sqrt{\tilde{m}|\tilde{n}|}}{c}\right).\end{aligned}$$

Setting $s_1=\sigma+\frac{it}{2}$ and $s_2=\sigma-\frac{it}{2}$ with $\sigma>1$ gives

$$\begin{aligned}&I_{m,n}\left(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2}\right)\\&\quad=\left(\frac{\tilde{m}}{|\tilde{n}|}\right)^{-\frac{it}{2}}\frac{2^{3-2\sigma}i^{-k}\pi\Gamma(2\sigma-1)}{\Gamma\left(\sigma-\frac{k}{2}+\frac{it}{2}\right)\Gamma\left(\sigma+\frac{k}{2}-\frac{it}{2}\right)}\sum_{c>0}\frac{S(m,n,c,\nu)}{c^{2\sigma}}K_{it}\left(\frac{4\pi\sqrt{\tilde{m}|\tilde{n}|}}{c}\right).\end{aligned}\tag{4.10}$$

To compute the inner product in the second way we introduce the notation

$$\Lambda(s_1,s_2,r)=\Gamma\left(s_1-\tfrac{1}{2}-ir\right)\Gamma\left(s_1-\tfrac{1}{2}+ir\right)\Gamma\left(s_2-\tfrac{1}{2}-ir\right)\Gamma\left(s_2-\tfrac{1}{2}+ir\right).$$

One has (see also [33, (32)])

$$\begin{aligned}&\xi_{\mathfrak{a}}(r,\,\mathcal{U}_m(\cdot,s,k,\nu))\\&\qquad=\overline{\varphi_{\mathfrak{a}m}\left(\tfrac{1}{2}+ir\right)}(4\pi\tilde{m})^{1-s}\tilde{m}^{-\frac{1}{2}-ir}e\left(\tfrac{k}{4}\right)\pi^{\frac{1}{2}-ir}\frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{\Gamma(s-\frac{k}{2})\Gamma(\frac{1}{2}+\frac{k}{2}-ir)}\end{aligned}$$

and

$$\begin{aligned}&\overline{\xi_{\mathfrak{a}}\left(r,\,\overline{\mathcal{U}_{1-n}(\cdot,s,-k,\overline{\nu})}\right)}\\&\qquad=\varphi_{\mathfrak{a}n}\left(\tfrac{1}{2}+ir\right)(4\pi|\tilde{n}|)^{1-s}|\tilde{n}|^{-\frac{1}{2}+ir}e\left(-\tfrac{k}{4}\right)\pi^{\frac{1}{2}+ir}\frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{\Gamma(s+\frac{k}{2})\Gamma(\frac{1}{2}-\frac{k}{2}+ir)}.\end{aligned}$$

Applying Parseval's identity (4.6) we get

$$I_{m,n}(s_1,s_2)=\frac{(4\pi)^{2-s_1-s_2}\tilde{m}^{1-s_1}|\tilde{n}|^{1-s_2}}{\Gamma\left(s_1-\frac{k}{2}\right)\Gamma\left(s_2+\frac{k}{2}\right)}\left(\sum_{r_j}\overline{\rho_j(m)}\rho_j(n)\Lambda(s_1,s_2,r_j)\right.$$
$$\left.+\sum_{\text{singular }\mathfrak{a}}\frac{1}{4\sqrt{\tilde{m}|\tilde{n}|}}\int_{-\infty}^{\infty}\left(\frac{|\tilde{n}|}{\tilde{m}}\right)^{ir}\frac{\overline{\varphi_{\mathfrak{a}m}\left(\frac12+ir\right)}\varphi_{\mathfrak{a}n}\left(\frac12+ir\right)\Lambda\left(s_1,s_2,r\right)}{\Gamma\left(\frac12+\frac{k}{2}-ir\right)\Gamma\left(\frac12-\frac{k}{2}+ir\right)}dr\right),$$

and for $s_1=\sigma+\frac{it}{2}$ and $s_2=\sigma-\frac{it}{2}$,

$$\begin{aligned}&I_{m,n}\left(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2}\right)\\&=\frac{(4\pi)^{2-2\sigma}|\tilde{m}\tilde{n}|^{1-\sigma}\left(\tilde{m}/|\tilde{n}|\right)^{-\frac{it}{2}}}{\Gamma\left(\sigma-\frac{k}{2}+\frac{it}{2}\right)\Gamma\left(\sigma+\frac{k}{2}-\frac{it}{2}\right)}\left(\sum_{r_j}\overline{\rho_j(m)}\rho_j(n)\Lambda\left(\sigma+\tfrac{it}{2},\sigma-\tfrac{it}{2},r\right)\right.\\&\left.+\sum_{\text{singular }\mathfrak{a}}\frac{1}{4\sqrt{\tilde{m}|\tilde{n}|}}\int_{-\infty}^{\infty}\left(\frac{|\tilde{n}|}{\tilde{m}}\right)^{ir}\frac{\overline{\varphi_{\mathfrak{a}m}\left(\frac12+ir\right)}\varphi_{\mathfrak{a}n}\left(\frac12+ir\right)\Lambda\left(\sigma+\frac{it}{2},\sigma-\frac{it}{2},r\right)}{\Gamma\left(\frac12+\frac{k}{2}-ir\right)\Gamma\left(\frac12-\frac{k}{2}+ir\right)}dr\right).\end{aligned}\tag{4.11}$$

When $\alpha_\nu=0(=\alpha_{\overline{\nu}})$, we define

$$I'_{m,n}(s_1,s_2):=\left\langle \mathcal{U}_m\left(\cdot,s_1,k,\nu\right),\overline{\mathcal{U}_{-n}\left(\cdot,s_2,-k,\overline{\nu}\right)}\right\rangle$$

and the same process above shows that $I'_{m,n}(s_1,s_2)$ equals the right hand side of both (4.10) and (4.11).

The expressions (4.10) and (4.11) are equal when $\sigma>1$ and we justify their equality when $\sigma=1$. The first expression (4.10) involving $K_{it}$ converges absolutely uniformly for $\sigma\in[1,2]$ because of [31, (10.45.7)]:

$$K_{it}(x)\ll(t\operatorname{sh}\pi t)^{-\frac12}\quad\text{as }x\to0$$

and condition (2) in Definition 1.6. By the basic inequalities

$$\left|\Gamma(\sigma-\tfrac12+iy)\right|=\frac{|\Gamma(\sigma+\frac12+iy)|}{|\sigma-\frac12+iy|}\le2\left|\Gamma(\sigma+\tfrac12+iy)\right|,$$

$\Lambda(\sigma+\frac{it}{2},\sigma-\frac{it}{2},y)>0$, and $|\rho\rho'|\le|\rho|^2+|\rho'|^2$ for all $y\in\mathbb{R}$ and $\rho,\rho'\in\mathbb{C}$, the second expression (4.11) involving $\Lambda$ also converges absolutely uniformly for $\sigma\in[1,2]$ as a result of its absolute convergence for $\sigma>1$.

With $\sigma=1$, we set (4.10) and (4.11) equal, cancel their common factors, multiply by

$$2\pi\sqrt{\tilde{m}|\tilde{n}|}\cdot\frac{2}{\pi^2}\,t\operatorname{sh}\pi t\int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}$$

and integrate over $t$. The following equations are helpful for getting (4.8):

(1) Kontorovich-Lebedev transform [45, (35)]

$$\frac{2}{\pi^2}\int_0^\infty K_{it}(x)t\operatorname{sh}\pi t\int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}\,dt=\frac{\phi(x)}{x};$$

(2) $\Lambda(1+\frac{it}{2},1-\frac{it}{2},r)\operatorname{ch}\pi(\frac{t}{2}+r)\operatorname{ch}\pi(\frac{t}{2}-r)=\pi^2$;

(3) [45, (39)]

$$\int_0^\infty \frac{t \operatorname{sh} \pi t}{\operatorname{ch} \pi(\frac{t}{2}+r) \operatorname{ch} \pi(\frac{t}{2}-r)} \int_0^\infty K_{it}(u)\phi(u)\frac{du}{u^2}\, dt = \frac{2}{\operatorname{ch} \pi r}\check{\phi}(r).$$

Theorem 4.11 now follows from this integration.

□

## 4.3 Estimating $\check{\phi}$ for the full spectrum and a special $\phi$

We focus on the case $k = \pm\frac{1}{2}$ and $\nu$ admissible as in Definition 1.6. Recall the notations $\Gamma = \Gamma_0(N)$, $a$, $T$, $\delta$, $\phi$ in Settings 4.8 and 4.9 and $\check{\phi}$ in (4.7). Recall that we write an eigenvalue $\lambda$ of $\Delta_k$ for $(\Gamma_0(N), \nu)$ as $\lambda = \frac{1}{4} + r^2$ where $r \in i(0, \frac{1}{4}] \cup [0, \infty)$. Bounds are known for $\check{\phi}(r)$ when $r \geq 1$ and here we give bounds for $r \in i(0, \frac{1}{4}] \cup [0, 1]$. For simplicity, we omit the dependence of the implied constants on $N$, $\nu$ and $\varepsilon$ in this section.

### 4.3.1 For $r \in i(0, \frac{1}{4}]$

Suppose $r = it$ for $t \in (0, \frac{1}{4}]$. By [31, (10.27.3), §10.37], for fixed $u$, $K_{-t}(u) = K_t(u) > 0$ is increasing as a function of $t > 0$. By [31, (10.7.7), (10.27.8)] we have

$$K_{2t}(u) \ll \frac{\Gamma(2t)2^{2t}}{u^{2t}}, \quad u \leq 1.$$

As for the discrete spectrum, there is a lower bound $\underline{t}$ for $t \in (0, \frac{1}{4}]$, hence an upper bound for $\Gamma(2t)$ depending on $N$. Thus,

$$K_{2t}(u) \ll \frac{1}{u^{2t}}, \quad u \leq 1.$$

By [31, (10.25.3)], we also have

$$K_{2t}(u) \ll e^{-u}, \quad u > 1.$$

Let $[\alpha, \beta] = \varnothing$ when $\beta < \alpha$. We get

$$\begin{aligned} \check{\phi}(it) &= \cos \pi t \int_0^\infty K_{2t}(u)\phi(u)\frac{du}{u} \\ &\ll \int_{[\frac{3a}{8x},1]} \frac{du}{u^{1+2t}} + O\left(\int_{[1,\frac{3a}{2x}]} e^{-u} du\right) \\ &\ll \left(\frac{x}{a}\right)^{2t} + O(1). \end{aligned} \tag{4.12}$$

In addition, assuming $H_\theta$ (2.15), by condition (1) in Definition 1.6 and Proposition 4.7, there are only two cases for $r = it$: $t = \frac{1}{4}$ or $2t \leq \theta$. In the second case, we have

$$\check{\phi}(r) \ll \left(\frac{x}{a}\right)^\theta + O(1), \quad r \neq \frac{i}{4}. \tag{4.13}$$

### 4.3.2 For real $|r| \in [0, 1)$

We cite [46, pp. 9.6.1, 9.8.5, 9.8.6] for numerical estimations of $K_0$:

1. $K_0(u) > 0$ for $u > 0$,
2. $K_0(u) \ll -\log\left(\frac{u}{2}\right)$ for $0 < u \le 2$,
3. $K_0(u) \ll u^{-\frac{1}{2}} e^{-u}$ for $u \ge 2$,

and have

$$\begin{aligned}\check{\phi}(0) &= \int_0^\infty K_0(u)\phi(u)\frac{du}{u} \\ &\ll \int_{[\frac{3a}{8x},2]} -\log\left(\frac{u}{2}\right)\frac{du}{u} + \int_{[2,\frac{3a}{2x}]} u^{-\frac{3}{2}} e^{-u} du \\ &\ll \left(\log\frac{3a}{16x}\right)^2 + e^{-2} \ll (ax)^{\varepsilon}.\end{aligned} \tag{4.14}$$

The last inequality is due to a positive lower bound for $a = 4\pi\sqrt{|\tilde{m}\tilde{n}|} \ge 4\pi \min(\alpha_\nu, 1-\alpha_\nu)$ when $\alpha_\nu > 0$ and $a \ge 4\pi$ when $\alpha_\nu = 0$, as $\tilde{m}\tilde{n} \ne 0$.

When $r \in (0,1)$, by [31, (10.32.9)]

$$|K_{ir}(u)| \le \int_0^\infty e^{-u \operatorname{ch} w} dw = K_0(u).$$

It follows from (4.14) that

$$\check{\phi}(r) \ll (ax)^{\varepsilon}, \quad r \in [0,1). \tag{4.15}$$

### 4.3.3 For $r \ge 1$

These bounds are recorded in [47, Theorem 5.1]. The first bound corrects [11, Theorem 6.1] and [12, Proposition 6.2] (but later estimates in their paper are not affected).

$$\check{\phi}(r) \ll \begin{cases} e^{-\frac{r}{2}} & \text{for } \ 1 \le r \le \frac{a}{8x}, \\ r^{-1} & \text{for } \ \max\left(1, \frac{a}{8x}\right) \le r \le \frac{a}{x}, \\ \min\left(r^{-\frac{3}{2}}, r^{-\frac{5}{2}}\frac{x}{T}\right) & \text{for } \ r \ge \max\left(\frac{a}{x}, 1\right). \end{cases} \tag{4.16}$$

### 4.3.4 A special test function

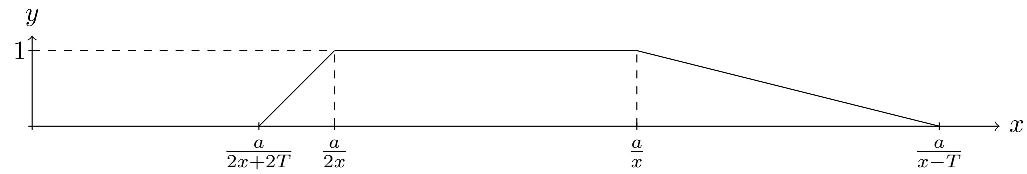
Here we choose a special test function $\phi$ satisfying Setting 4.9 to compute the terms corresponding to the exceptional spectrum $r \in i(0, \frac{1}{4}]$ in Theorem 1.7.

For a general weight $k > 0$ and $\tilde{m} > 0$, $\tilde{n} < 0$ with exceptional eigenvalue $\lambda < \frac{1}{4}$, we set $\lambda = s(1-s)$ for $s \in (\frac{1}{2}, 1)$ and

$$t = \operatorname{Im} r = \sqrt{\tfrac{1}{4} - \lambda} = \sqrt{\tfrac{1}{4} - s(1-s)} = s - \tfrac{1}{2}.$$

In (4.12) the exponent is $2t = 2s-1$. Let the lower bound for $t > 0$ be $\underline{t}$ depending on $N$ and $0 < T' \le T \le \frac{x}{3}$ be $T' := Tx^{-\delta} \asymp x^{1-2\delta}$.

**Setting 4.12.** *In addition to the requirement in Definition 4.9, when $\frac{a}{x-T} \le 1.999$, we pick $\phi$ as a smoothed function of this piecewise linear one*




*where*

$$
\begin{cases}
\phi'(u) = \frac{2x(x+T)}{aT} & u \in (\frac{a}{2x+2T-2T'}, \frac{a}{2x+2T'}), \\
\phi'(u) = -\frac{x(x-T)}{aT} & u \in (\frac{a}{x-T'}, \frac{a}{x-T+T'}), \\
0 \le \phi'(u) \le \frac{4x(x+T)}{aT} & u \in (\frac{a}{2x+2T}, \frac{a}{2x+2T-2T'}) \cup (\frac{a}{2x+2T'}, \frac{a}{2x}), \\
0 \ge \phi'(u) \ge -\frac{2x(x-T)}{aT} & u \in (\frac{a}{x}, \frac{a}{x-T'}) \cup (\frac{a}{x-T+T'}, \frac{a}{x-T}), \\
\phi'(u) = 0 & \textit{otherwise.}
\end{cases}
\tag{4.17}
$$

The above choice for $\phi'$ is possible because there is no requirement for $\phi''(u)$ when $u \le 2$ but for $u \to \infty$ in Setting 4.9.

Derived from [31, (10.25.2), (10.27.4), (10.37.1)], for $r = it$ and $2t \in [2\underline{t}, \frac{1}{2}]$, we have

$$K_{2t}(u) = 2^{2t-1}\Gamma(2t)u^{-2t} + O((\tfrac{u}{2})^{2t}) \quad \text{uniformly for } |u| \le 1.999$$

and

$$|K_{2t}(u)| \le |K_{\frac{1}{2}}(u)| \ll u^{-\frac{1}{2}}e^{-u} \qquad \text{uniformly for } u \ge 1.$$

Thus, for $r = it \in i[\underline{t}, \frac{1}{4}]$,

$$
\begin{aligned}
\frac{1}{\cos \pi t}\check{\phi}(r) &= 2^{2t-1}\Gamma(2t)\int_0^{1.999} \phi(u)u^{-2t}\frac{du}{u} + O\left(\int_0^{1.999} \phi(u)u^{2t}\frac{du}{u}\right) + O(1) \\
&= 2^{2t-1}\Gamma(2t)\int_0^{1.999} \phi(u)u^{-2t}\frac{du}{u} + O(1).
\end{aligned}
\tag{4.18}
$$

**Lemma 4.13.** *With the choice of $\phi$ in Setting 4.12, when $r = it \in i(0, \frac{1}{4}]$,*

$$
\begin{aligned}
\frac{1}{\cos \pi t}\check{\phi}(r) &= 2^{2t-1}\Gamma(2t)\int_{\frac{a}{2x}}^{\frac{a}{x}} u^{-2t-1}du + O\left(x^{2t-\delta}a^{-2t} + 1\right) \\
&= \frac{2^{2t-1}(2^{2t}-1)}{2t}\Gamma(2t)\left(\frac{x}{a}\right)^{2t} + O\left(x^{2t-\delta}a^{-2t} + 1\right).
\end{aligned}
\tag{4.19}
$$

Roughly speaking, this means that the integral on $u \in (\frac{a}{2x}, \frac{a}{x})$ contributes the main term when $x$ is large.

*Proof of Lemma 4.13.* When $1.999 < \frac{a}{x-T} \le \frac{3a}{2x}$, we get $x \ll a$ and $\check{\phi}(r) = O(1)$ by (4.12), so the lemma is true in this case. It suffices to prove that when $\frac{a}{x-T} \le 1.999$, the integral on $u < \frac{a}{2x}$ and $u > \frac{a}{x}$ is $O(x^{2t-\delta}a^{-2t})$. As $T \asymp x^{1-\delta}$ where $\delta > 0$, we apply

$$\left(1 \pm \alpha x^{-\delta}\right)^{2t-1} = 1 \pm (2t-1)\alpha x^{-\delta} + O(x^{-2\delta}) = 1 + O(x^{-\delta})$$

where the implied constants are absolute since $2t-1\in(-1,-\frac{1}{2}]$. Combining the lower bound $t\geq \underline{t}$ depending on $N$, for the left part $u\in[\frac{a}{2x+2T},\frac{a}{2x}]$, we use (4.17) and the above formula to compute the integral in (4.18):

$$\begin{aligned}\int_{\frac{a}{2x+2T}}^{\frac{a}{2x}} u^{-2t-1}\phi(u)du &= -\frac{2^{2t}}{2t}\left(\frac{x}{a}\right)^{2t}+\frac{2x(x+T)}{2taT}\int_{\frac{a}{2x+2T}}^{\frac{a}{2x}} u^{-2t}du\\ &+O\left(\frac{x^{1+\delta}}{at}\right)\left(\int_{\frac{a}{2x+2T}}^{\frac{a}{2x+2T-2T'}}+\int_{\frac{a}{2x+2T'}}^{\frac{a}{2x}}\right)u^{-2t}du\\ &=-\frac{2^{2t}}{2t}\left(\frac{x}{a}\right)^{2t}+\frac{2x(x+T)}{2taT(1-2t)}\left(\frac{a}{2x}\right)^{1-2t}\left((1-2t)\frac{T}{x}+O(x^{-2\delta})\right)\\ &+O\left(\frac{x^{1+\delta}}{at}\right)\left(\left(\frac{a}{2x+2T}\right)^{1-2t}+\left(\frac{a}{2x}\right)^{1-2t}\right)O(x^{-2\delta})\\ &=-\frac{2^{2t}}{2t}\left(\frac{x}{a}\right)^{2t}+\frac{2^{2t}(1+\frac{T}{x})}{2t}\left(\frac{x}{a}\right)^{2t}+O\left(x^{2t-\delta}a^{-2t}\right)\\ &=O\left(x^{2t-\delta}a^{-2t}\right).\end{aligned}$$

For $u\in[\frac{a}{x},\frac{a}{x-T}]$, a similar process gives the same conclusion. □

## 4.4 Two estimates for the coefficients of Maass forms

Our main results depend on two estimates for the Fourier coefficients of Maass forms. These estimates were recorded in [12, Section 4] but only for the coefficients of Maass cusp forms. Here we also require estimates for the coefficients of Eisenstein series.

Recall our notations in Settings 4.8 and 4.9. In [44] an estimate for the coefficients of Maass cusp forms was given under the hypothesis that for some $\beta\in(\frac{1}{2},1)$,

$$\sum_{N|c>0}\frac{|S(n,n,c,\nu)|}{c^{1+\beta}}\ll_{\varepsilon,\nu}|\tilde{n}|^{\varepsilon}. \tag{4.20}$$

Here we prove

**Proposition 4.14.** *Suppose that $\nu$ is a multiplier on $\Gamma=\Gamma_0(N)$ of weight $k=\pm\frac{1}{2}$ which satisfies (4.20). Let $\rho_j(n)$ denote the Fourier coefficients of an orthonormal basis $\{v_j(\cdot)\}$ of $\tilde{\mathcal{L}}_k(N,\nu)$. For each singular cusp $\mathfrak{a}$ of $(\Gamma,\nu)$, let $E_{\mathfrak{a}}(\cdot,s)$ be the associated Eisenstein series. Let $\varphi_{\mathfrak{a}n}(\frac{1}{2}+ir)$ and $\rho_{\mathfrak{a}}(n,r)$ be defined as in (4.3). Then for $x>0$ we have*

$$\begin{aligned}x^{k\,\mathrm{sgn}\,\tilde{n}}|\tilde{n}|\left(\sum_{x\leq r_j\leq 2x}|\rho_j(n)|^2e^{-\pi r_j}+\sum_{\text{singular }\mathfrak{a}}\int_{|r|\in[x,2x]}|\rho_{\mathfrak{a}}(n,r)|^2e^{-\pi|r|}dr\right)&\\ \ll_{\varepsilon,N} x^2+|\tilde{n}|^{\beta+\varepsilon}x^{1-2\beta}\log^{\beta}x.&\end{aligned}$$

*Remark.* Since we focus on an admissible multiplier, condition (2) in Definition 1.6 allows us to choose $\beta=\frac{1}{2}+\varepsilon$ when applying this proposition.

*Proof.* First we suppose $\tilde{n}>0$. The proof follows the same argument as [44, Section 4]. We note that the coefficients of the Eisenstein series are not normalized correctly in their Lemma 4.2 (check with [33, Lemma 3]). This does not affect [44, Theorem 4.1] since these terms were dropped by positivity.

With the assumptions of Proposition 4.14, for any $t \in \mathbb{R}$, using (4.3) and (4.9), the correct lemma is

$$\begin{aligned}\frac{2\pi^2\tilde{n}}{|\Gamma(1-\frac{k}{2}+it)|^2}&\left(\sum_{r_j}\frac{|\rho_j(n)|^2}{\operatorname{ch}2\pi r_j+\operatorname{ch}2\pi t}+\frac{1}{4\pi}\sum_{\text{singular }\mathfrak{a}}\int_{-\infty}^{\infty}\frac{|\rho_{\mathfrak{a}}(n,r)|^2dr}{\operatorname{ch}2\pi r+\operatorname{ch}2\pi t}\right)\\&=\frac{1}{4\pi}+\frac{2\tilde{n}}{i^{k+1}}\sum_{N|c>0}\frac{S(n,n,c,\nu)}{c^2}\int_L K_{2it}\left(\frac{4\pi\tilde{n}}{c}q\right)q^{k-1}dq,\end{aligned}$$

where $L$ is the semicircular contour $|q|=1$ with $\operatorname{Re} q>0$ from $-i$ to $i$. Let $K$ be a large positive real number. Using [44, Lemma 4.3] we get the full version of [44, (4.3)]:

$$\begin{aligned}\tilde{n}&\left(\sum_{r_j}|\rho_j(n)|^2h_K(r_j)+\sum_{\text{singular }\mathfrak{a}}\int_{-\infty}^{\infty}|\rho_{\mathfrak{a}}(n,r)|^2h_K(r)dr\right)\\&\ll K+\sum_{N|c>0}\frac{|S(n,n,c,\nu)|}{c}\left|M_k\left(K,\frac{2\pi\tilde{n}}{c}\right)\right|\end{aligned}\tag{4.21}$$

where

$$h_K(r):=\int_{-\infty}^{\infty}\frac{e^{-(t/K)^2}-e^{-(2t/K)^2}}{|\Gamma(1-\frac{k}{2}+it)|^2|(\operatorname{ch}2\pi r+\operatorname{ch}2\pi t)}dt$$

and

$$M(K,\alpha)=\int_{-\infty}^{\infty}\left(e^{-(t/K)^2}-e^{-(2t/K)^2}\right)\int_{(\xi)}\frac{\sin(\pi s-\frac{\pi k}{2})}{s-\frac{k}{2}}\Gamma(s+it)\Gamma(s-it)\alpha^{1-2s}dsdt.$$

The right hand side of (4.21) is estimated in [44, Section 4] where we get

$$\begin{aligned}\tilde{n}&\left(\sum_{r_j}|\rho_j(n)|^2h_K(r_j)+\sum_{\text{singular }\mathfrak{a}}\int_{-\infty}^{\infty}|\rho_{\mathfrak{a}}(n,r)|^2h_K(r)dr\right)\\&=O_{\varepsilon,N}\left(x+\tilde{n}^{\beta+\varepsilon}x^{-2\beta}\log^{\beta}x\right).\end{aligned}\tag{4.22}$$

Observe that $h_K(r)$ is even as a function of $r$ and $h_x(r)\gg e^{-\pi|r|}x^{k-1}$ when $|r|\asymp x$. This proves Proposition 4.14 when $\tilde{n}>0$.

The $\tilde{n}<0$ case follows from conjugation by (1.13), (1.14) and (2.17), which is similar to [12, Section 4]. ☐

We also require a generalization of [12, Theorem 4.3] which includes the contribution from Eisenstein series.

**Proposition 4.15.** *Let $M$ be a positive integer which is a multiple of 4. Let*

$$(k,\nu')=(\tfrac{1}{2},(\tfrac{|D|}{\cdot})\nu_\theta)\quad\text{or}\quad(-\tfrac{1}{2},(\tfrac{-|D|}{\cdot})\nu_\theta)=(-\tfrac{1}{2},(\tfrac{|D|}{\cdot})\overline{\nu_\theta}),$$

*where $D$ is an even fundamental discriminant dividing $M$. Suppose that $\nu$ is a weight $k$ admissible (Definition 1.6) multiplier on $\Gamma=\Gamma_0(N)$ with $M$, $D$, $\nu'$ above for some integer $B>0$. Let $\rho_j(n)$ denote the Fourier coefficients of an orthonormal basis $\{v_j(\cdot)\}$ of $\tilde{\mathcal{L}}_k(N,\nu)$. For each singular cusp $\mathfrak{a}$, let $\rho_{\mathfrak{a}}(n,r)$ be defined as in (4.3) corresponding to the Eisenstein series on $(\Gamma_0(N),\nu)$. Suppose $x\geq 1$.*

*For $n\neq 0$ square-free or coprime to $M$ we have*

$$x^{k\operatorname{sgn}\tilde n}|\tilde n|\left(\sum_{|r_j|\le x}\frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j}+\sum_{\text{singular }\mathfrak a}\int_{-x}^{x}\frac{|\rho_{\mathfrak a}(n,r)|^2}{\operatorname{ch}\pi r}dr\right)\ll_{\nu,\varepsilon}|\tilde n|^{\frac{131}{294}+\varepsilon}x^3.$$

*In general, for $n\neq 0$ we factor $B\tilde n=t_nu_n^2w_n^2$ where $t_n$ is square-free, $u_n|M^\infty$ and $(w_n,M)=1$. Then we have*

$$x^{k\operatorname{sgn}\tilde n}|\tilde n|\left(\sum_{|r_j|\le x}\frac{|\rho_j(n)|^2}{\operatorname{ch}\pi r_j}+\sum_{\text{singular }\mathfrak a}\int_{-x}^{x}\frac{|\rho_{\mathfrak a}(n,r)|^2}{\operatorname{ch}\pi r}dr\right)\ll_{\nu,\varepsilon}\left(|\tilde n|^{\frac{131}{294}}+u_n\right)x^3|\tilde n|^{\varepsilon}.$$

The proof of Proposition 4.15 uses Iwaniec's averaging method as in [12]. One important property is the relationship between the Fourier coefficients in different levels. This is not hard via the inner product for Maass cusp forms, but not clear for the continuous spectrum. Here we apply arguments in [48] for the calculations.

For the remaining part of this section we identify the levels. Let $\langle\cdot,\cdot\rangle_{(N)}$ denote the Petersson inner product over the fundamental domain $\Gamma_0(N)\setminus\mathbb H$. For integer $q\ge 1$, let $w_q:=\left(\begin{smallmatrix}\sqrt q&0\\0&1/\sqrt q\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb R)$.

Suppose that $\nu^{(S)}$ is a weight $k=\pm\frac12$ multiplier on $\Gamma_0(S)$ and $\nu^{(T)}$ is a weight $k$ multiplier on $\Gamma_0(T)$. Suppose that there exist positive integers $q$ and $T$ such that

$$f(z)\in\mathcal A_k(S,\nu^{(S)})\quad\Rightarrow\quad f(qz)=(f|_kw_q)(z)\in\mathcal A_k(T,\nu^{(T)}).\tag{4.23}$$

Note that this relation implies $qS|T$.

For $L\in\{S,T\}$, let $\rho_j^{(L)}(n)$ denote the Fourier coefficients of an orthonormal basis $\{v_j^{(L)}(\cdot)\}$ of $\tilde{\mathcal L}_k(L,\nu^{(L)})$. For each singular cusp $\mathfrak a$ of $(\Gamma_0(L),\nu^{(L)})$, let $E_{\mathfrak a}^{(L)}(\cdot,s)$ be the associated Eisenstein series. Let $\varphi_{\mathfrak a n}^{(L)}(\frac12+ir)$ and $\rho_{\mathfrak a}^{(L)}(n,r)$ be defined as in (4.3).

Let $\mathcal V_j^{(T)}(z)=v_j^{(S)}(qz)$ and $\mathcal E_{\mathfrak a}^{(T)}(z,s)=E_{\mathfrak a}^{(S)}(qz,s)$. So $\mathcal V_j^{(T)}$ and $\mathcal E_{\mathfrak a}^{(T)}(\cdot,\frac12+ir)$ are eigenfunctions corresponding to the discrete and continuous spectrum of $\Delta_k$, respectively. Let $n_{(L)}:=n-\alpha_{\nu^{(L)}}$ for $n\in\mathbb Z$ and suppose $\alpha_{\nu^{(T)}}=0$. Then $qn_{(S)}\in\mathbb Z$ and

$$\rho_j^{(S)}(n)=\mathcal P_j^{(T)}(qn_{(S)})\quad\text{and}\quad\rho_{\mathfrak a}^{(S)}(n,r)=\mathcal P_{\mathfrak a}^{(T)}(qn_{(S)},r),\tag{4.24}$$

where $\mathcal P_j^{(T)}(n)$ and $\mathcal P_{\mathfrak a}^{(T)}(n,r)$ are the Fourier coefficients of $\mathcal V_j^{(T)}$ and $\mathcal E_{\mathfrak a}^{(T)}(\cdot,\frac12+ir)$ as in (4.3), respectively.

Since $d\mu(z)=\frac{dxdy}{y^2}$ is invariant under $\mathrm{GL}_2^+(\mathbb R)$, we can denote $I(S,T)$ as the normalizing constant such that

$$\langle f(q\cdot),g(q\cdot)\rangle_{(T)}=I(S,T)\langle f,g\rangle_{(S)},\quad\text{for all } f,g\in\mathcal L_k(S,\nu^{(S)}).$$

So $I(S,T)$ is the index $[\Gamma_0(S):\Gamma_0(T)]$. The set

$$\left\{I(S,T)^{-\frac12}\mathcal V_j^{(T)}(\cdot): r_j\text{ of }\Gamma_0(S)\right\}$$

is an orthonormal subset in $\tilde{\mathcal L}_k(T,\nu^{(T)})$ and can be expand to a orthonormal basis of $\tilde{\mathcal L}_k(T,\nu^{(T)})$ as

$$\left\{\frac{\mathcal V_j^{(T)}(\cdot)}{I(S,T)^{\frac12}}: r_j\text{ of }\Gamma_0(S)\right\}\bigcup\left\{w_j^{(T)}(\cdot): r_j\text{ of }\Gamma_0(T)\right\},\tag{4.25}$$

where each $w_j^{(T)}$ is a linear combination of $v_j^{(T)}$ from the standard basis. Let the Fourier coefficients of $w_j$ be denoted as $\rho_j^{\text{comp}}(n)$, which is the corresponding linear combination of $\{\rho_j^{(T)}(n)\}_j$.

For the continuous spectrum, as in [48, (8.1)] we let $\mathscr{E}_r(L)$ be the finite dimensional space

$$\mathscr{E}_r(L) := \operatorname{span}\{E_{\mathfrak{a}}^{(L)}(\cdot, \tfrac{1}{2}+ir):\ \text{singular } \mathfrak{a} \text{ of } (\Gamma_0(L), \nu^{(L)})\}.$$

This $\mathscr{E}_r(L)$ is also the subspace of eigenfunctions in the continuous spectrum of $\Delta_k$ with eigenvalue $\lambda = \frac{1}{4}+r^2$. We define the formal inner product

$$\langle \cdot,\cdot\rangle_{(L)}^{\text{Eis}}: \quad \Big\langle E_{\mathfrak{a}}^{(L)}(\cdot, \tfrac{1}{2}+ir),\ E_{\mathfrak{b}}^{(L)}(\cdot, \tfrac{1}{2}+ir)\Big\rangle_{(L)}^{\text{Eis}} = 4\pi\delta_{\mathfrak{a}\mathfrak{b}}^{(L)}, \tag{4.26}$$

where $\delta_{\mathfrak{a}\mathfrak{b}}^{(L)} = 1$ if cusps $\mathfrak{a}$ and $\mathfrak{b}$ are $\Gamma_0(L)$-equivalent and $\delta_{\mathfrak{a}\mathfrak{b}}^{(L)} = 0$ otherwise. This inner product is extended sesquilinearly as a inner product on $\mathscr{E}_r(L)$, which means it is conjugate linear at the first entry and linear at the second entry.

Recall (4.4) for the Fourier expansion of Eisenstein series at the cusps. Since $\mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \frac{1}{2}+ir) = E_{\mathfrak{a}}^{(S)}(q\cdot, \frac{1}{2}+ir) \in \mathscr{E}_r(T)$ where $\mathfrak{a}$ is a singular cusp of $\Gamma_0(S)$, we can write

$$\mathcal{E}_{\mathfrak{a}}^{(T)}(z, \tfrac{1}{2}+ir) = \sum_{\substack{\text{singular } \mathfrak{b}\\ \text{of } \Gamma_0(T)}} c_{\mathfrak{a}}(\mathfrak{b}) E_{\mathfrak{b}}^{(T)}(z, \tfrac{1}{2}+ir). \tag{4.27}$$

Let

$$I(S,T,\mathfrak{a}) := \sum_{\substack{\text{singular } \mathfrak{b}\\ \text{of } \Gamma_0(T)}} |c_{\mathfrak{a}}(\mathfrak{b})|^2,$$

then we have

$$\Big\langle \mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir),\ \mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir)\Big\rangle_{(T)}^{\text{Eis}} = 4\pi I(S,T,\mathfrak{a}).$$

On the other hand, we know that $\mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \frac{1}{2}+ir) = E_{\mathfrak{a}}^{(S)}(\cdot, \frac{1}{2}+ir)|_k w_q$. Then the Fourier expansion of $\mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \frac{1}{2}+ir)$ at the cusp $\mathfrak{b}$ has a non-zero $y^{\frac{1}{2}+ir}$ term, if, and only if, the Fourier expansion of $E_{\mathfrak{a}}^{(S)}(\cdot, \frac{1}{2}+ir)$ at the cusp $w_q\mathfrak{b}$ has a non-zero $y^{\frac{1}{2}+ir}$ term. Since $E_{\mathfrak{b}}^{(T)}(\cdot, \frac{1}{2}+ir)$ only has non-zero $y^{\frac{1}{2}+ir}$ term at the cusps equivalent to $\mathfrak{b}$ on $\Gamma_0(T)$, we can rewrite (4.27) as

$$\mathcal{E}_{\mathfrak{a}}^{(T)}(z, \tfrac{1}{2}+ir) = \sum_{\substack{\text{singular } \mathfrak{b} \text{ of } \Gamma_0(T)\\ w_q\mathfrak{b} \text{ equivalent to } \mathfrak{a} \text{ on } \Gamma_0(S)}} c_{\mathfrak{a}}(\mathfrak{b}) E_{\mathfrak{b}}^{(T)}(z, \tfrac{1}{2}+ir). \tag{4.28}$$

The above sum is well defined. In fact, if two cusps $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are nonequivalent on $\Gamma_0(S)$, then $w_q^{-1}\mathfrak{a}_1$ and $w_q^{-1}\mathfrak{a}_2$ are nonequivalent on $\Gamma_0(T)$. This is easily verified with (4.23) by $qS|T$ and

$$\gamma^{(T)} \in \Gamma_0(T) \quad \Rightarrow \quad w_q\gamma^{(T)}w_q^{-1} \in \Gamma_0(S).$$

Then the sums in (4.28) for $\mathcal{E}_{\mathfrak{a}_1}^{(T)}$ and $\mathcal{E}_{\mathfrak{a}_2}^{(T)}$ are on disjoint singular cusps of $\Gamma_0(T)$.

Therefore, by the orthogonality in $\mathcal{E}_r(T)$ with respect to $\langle\cdot,\cdot\rangle_{(T)}^{\text{Eis}}$,

$$\Big\langle \mathcal{E}_{\mathfrak{a}_1}^{(T)}(\cdot, \tfrac{1}{2}+ir),\ \mathcal{E}_{\mathfrak{a}_2}^{(T)}(\cdot, \tfrac{1}{2}+ir)\Big\rangle_{(T)}^{\text{Eis}} = 4\pi I(S,T,\mathfrak{a}_1)\delta_{\mathfrak{a}_1\mathfrak{a}_2}^{(S)}. \tag{4.29}$$

Now we can expand the set

$$\left\{ I(S,T,\mathfrak{a})^{-\frac{1}{2}} \mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir) : \text{ singular } \mathfrak{a} \text{ of } \Gamma_0(S) \right\}$$

to an orthonormal basis of $\mathcal{E}_r(T)$ with respect to $\langle \cdot, \cdot \rangle_{(T)}^{\mathrm{Eis}}$ as

$$\left\{ \frac{\mathcal{E}_{\mathfrak{a}}^{(T)}(\cdot, \frac{1}{2}+ir)}{I(S,T,\mathfrak{a})^{\frac{1}{2}}} : \text{ singular } \mathfrak{a} \text{ of } \Gamma_0(S) \right\} \bigcup \left\{ F_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir) : \text{ singular } \mathfrak{a} \text{ of } \Gamma_0(T) \right\}, \tag{4.30}$$

where each $F_{\mathfrak{a}}^{(T)}$ is some linear combination of $E_{\mathfrak{a}}^{(T)}$. We denote the Fourier coefficients of $F_{\mathfrak{a}}^{(T)}$ as $\rho_{\mathfrak{a}}^{\mathrm{comp}}(n,r)$ and $\varphi_{\mathfrak{a}n}^{\mathrm{comp}}(\frac{1}{2}+ir)$ as (4.3), which are corresponding linear combinations of $\rho_{\mathfrak{a}}^{(T)}(n,r)$ or $\varphi_{\mathfrak{a}n}^{(T)}(\frac{1}{2}+ir)$.

Recall the standard expansion for $h \in \mathcal{B}_k(T, \nu^{(T)})$ [35, Theorem 2.1]

$$\begin{aligned} h(z) &= \sum_{r_j \text{ of } \Gamma_0(T)} \langle h, v_j^{(T)} \rangle_{(T)} v_j^{(T)} + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(T)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle h, E_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir) \right\rangle_{(T)} E_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir) dr \\ &=: h_{\mathrm{D}}(z) + h_{\mathrm{C}}(z) \end{aligned} \tag{4.31}$$

For the discrete spectrum, we have an alternative orthonormal basis (4.25) hence another expansion for $h_{\mathrm{D}}(z)$. For the continuous spectrum, [48, Proposition 8.2] ensures that the above expansion fo $h_{\mathrm{C}}(z)$ is invariant with an alternative basis (4.30) (where we write Young's notation $\langle F, F \rangle_{\mathrm{Eis}} = 4\pi$ explicitly here to be consistent with our notations). Now we can deduce another expansion for $h$:

$$\begin{aligned} h(z) &= h_{\mathrm{D}}(z) + h_{\mathrm{C}}(z) \\ &= \sum_{r_j \text{ of } \Gamma_0(S)} \left\langle h, \frac{v_j(q\cdot)}{I(S,T)^{\frac{1}{2}}} \right\rangle_{(T)} \frac{v_j(qz)}{I(S,T)^{\frac{1}{2}}} + \sum_{r_j \text{ of } \Gamma_0(T)} \langle h, w_j \rangle_{(T)} w_j(z) \\ &+ \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(S)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle h, \frac{E_{\mathfrak{a}}^{(S)}\left(q\cdot, \frac{1}{2}+ir\right)}{I(S,T,\mathfrak{a})^{\frac{1}{2}}} \right\rangle_{(T)} \frac{E_{\mathfrak{a}}^{(S)}\left(qz, \frac{1}{2}+ir\right)}{I(S,T,\mathfrak{a})^{\frac{1}{2}}} dr \\ &+ \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(T)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle h, F_{\mathfrak{a}}^{(T)}(\cdot, \tfrac{1}{2}+ir) \right\rangle_{(T)} F_{\mathfrak{a}}^{(T)}(z, \tfrac{1}{2}+ir) dr. \end{aligned} \tag{4.32}$$

We now show that $I(S,T,\mathfrak{a}) = I(S,T)$. Let $h \in \mathcal{B}_k(S, \nu^{(S)})$ be orthogonal to the discrete spectrum, i.e. $h_{\mathrm{D}} = 0$. The standard spectral expansion of $h$ at level $S$ gives

$$h(z) = h_{\mathrm{C}}(z) = \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(S)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle h, E_{\mathfrak{a}}^{(S)}\left(\cdot, \tfrac{1}{2}+ir\right) \right\rangle_{(S)} E_{\mathfrak{a}}^{(S)}\left(z, \tfrac{1}{2}+ir\right) dr.$$

Especially,

$$h(qz) = \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(S)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \left\langle h, E_{\mathfrak{a}}^{(S)}\left(\cdot, \tfrac{1}{2}+ir\right) \right\rangle_{(S)} E_{\mathfrak{a}}^{(S)}\left(qz, \tfrac{1}{2}+ir\right) dr. \tag{4.33}$$

However, $H(\cdot) = h(q\cdot)$ is in $\mathcal{B}_k(T, \nu^{(T)})$ (and is still orthogonal to the discrete spectrum) with spectral expansion $H_{\mathrm{C}}(z)$ as (4.32). As we have shown the orthogonality of (4.30) in $\mathscr{E}_r(T)$ under $\langle \cdot, \cdot \rangle_{\mathrm{Eis}}^{(T)}$, the spectral expansion of $H(\cdot)$ has to be unique on a subset of the basis (4.30). Comparing (4.33) and (4.32) we have

$$\left\langle h(q\cdot), \frac{E_{\mathfrak{a}}^{(S)}\left(q\cdot, \frac{1}{2}+ir\right)}{I(S,T,\mathfrak{a})} \right\rangle_{(T)} = \langle h, E_{\mathfrak{a}}^{(S)}\left(\cdot, \tfrac{1}{2}+ir\right)\rangle_{(S)} \quad \Rightarrow \quad I(S,T,\mathfrak{a}) = I(S,T).$$

Now we are ready to state the formula connecting the Fourier coefficients of different level eigenforms. For $\sigma = \operatorname{Re} s > 1$, $t \in \mathbb{R}$ and $n > 0$, we compute the inner product

$$\left\langle U_{qn_{(S)}}^{(T)}(\cdot, \sigma + \tfrac{it}{2}), U_{qn_{(S)}}^{(T)}\left(\cdot, \sigma - \tfrac{it}{2}\right) \right\rangle_{(T)} \tag{4.34}$$

as [33, Lemma 2]. The results are the same if we apply the above decomposition (4.32) for $h(z) = U_{qn_{(S)}}^{(T)}(z, \sigma \pm \frac{it}{2})$ and if we apply the standard spectral decomposition (4.31). Recall the notation

$$\Lambda(\sigma+it, \sigma-it, r) = \left|\Gamma\left(\sigma - \tfrac{1}{2} + i(t+r)\right)\right|^2 \left|\Gamma\left(\sigma - \tfrac{1}{2} + i(t-r)\right)\right|^2.$$

For $\sigma > 1$, we can get two results of (4.34), as on the right hand side of [33, (30)], by the two different expansions mentioned above. The following equation is the identity between such two results, where we recall (4.24) for the relation in Fourier coefficients:

$$\begin{aligned}
&\sum_{r_j \text{ of } \Gamma_0(S)} I(S,T)^{-1} |\rho_j^{(S)}(n)|^2 \Lambda\left(\sigma + \tfrac{it}{2}, \sigma - \tfrac{it}{2}, r_j\right) \\
&+ \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(S)}} I(S,T)^{-1} \int_{-\infty}^{\infty} \frac{|\varphi_{\mathfrak{a}n}^{(S)}(\frac{1}{2}+ir)|^2 \Lambda\left(\sigma + \frac{it}{2}, \sigma - \frac{it}{2}, r\right) dr}{4n\, \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right) \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right)} \\
&\qquad + \sum_{r_j \text{ of } \Gamma_0(T)} |\rho_j^{\text{comp}}(qn_{(S)})|^2 \Lambda\left(\sigma + \tfrac{it}{2}, \sigma - \tfrac{it}{2}, r_j\right) \\
&\qquad + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(T)}} \int_{-\infty}^{\infty} \frac{|\varphi_{\mathfrak{a},qn_{(S)}}^{\text{comp}}(\frac{1}{2}+ir)|^2 \Lambda\left(\sigma + \frac{it}{2}, \sigma - \frac{it}{2}, r\right) dr}{4qn_{(S)}\, \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right) \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right)} \\
= &\sum_{r_j \text{ of } \Gamma_0(T)} |\rho_j^{(T)}(qn_{(S)})|^2 \Lambda\left(\sigma + \tfrac{it}{2}, \sigma - \tfrac{it}{2}, r_j\right) \\
&\qquad + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(T)}} \int_{-\infty}^{\infty} \frac{|\varphi_{\mathfrak{a},qn_{(S)}}^{(T)}(\frac{1}{2}+ir)|^2 \Lambda\left(\sigma + \frac{it}{2}, \sigma - \frac{it}{2}, r\right) dr}{4qn_{(S)}\, \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right) \Gamma\left(\frac{1}{2} + \frac{k}{2} - ir\right)}
\end{aligned} \tag{4.35}$$

With the help of (4.9) on the notations, we have proved a lemma regarding the shifting of levels:

**Lemma 4.16.** *Given $T > S > 0$ and $S|T$, with the notations above, for all $t \in \mathbb{R}$ and $\sigma > 1$ we have (4.35). In addition, with (4.9) we can also write the terms involving $\varphi_{\mathfrak{a}\ell}(\frac{1}{2}+ir)$ as those with $\rho_{\mathfrak{a}}(\ell, r)$ and we omit the duplicated formula here.*

### 4.4.1 Proof of Proposition 4.15

We still use superscript $\cdot^{(N)}$ to identify the level and should be careful on it. The notations $\rho_j$ and $\rho_{\mathfrak{a}}$ in the statement of Proposition 4.15 are on level $N$ and among the proof we utilize two more different levels. When

$B\tilde{n}$ is square-free, [12, Theorem 4.3] gives the bound

$$x^{k\,\mathrm{sgn}\,\tilde{n}}|\tilde{n}|\sum_{|r_j|\le x}\frac{|\rho_j^{(N)}(n)|^2}{\mathrm{ch}\,\pi r_j}\ll_{\nu,\varepsilon}|\tilde{n}|^{\frac{131}{294}+\varepsilon}x^3. \tag{4.36}$$

Our proposition generalizes the above bound. It suffices to prove the general case involving $u_n$ with $\tilde{n}>0$ and $k=\pm\frac{1}{2}$, where the $\tilde{n}<0$ case follows from conjugation by (1.14) and (2.17).

Following the notation in [12, §5.2], we can take the fundamental discriminant $D$ to be even and $M\equiv 0 \pmod 8$ as a positive integer with $D|M$. Let $P$ be a positive parameter (chosen later to be $n^{\frac{1}{7}}$) and

$$\mathcal{Q}=\mathcal{Q}(n,M,P):=\{pM:\ p \text{ prime},\ P<p\le 2P,\ \text{and } p\nmid 2nM\}.$$

We take any $pM$ in $\mathcal{Q}$. In [12, p.1698], they require the property that when $\{v_j^{(M)}\}$ is an orthonormal subset of $\tilde{\mathcal{L}}_k(M,\nu')$, then $\{[\Gamma_0(M):\Gamma_0(pM)]^{-\frac{1}{2}}v_j^{(M)}\}$ is an orthonormal subset of $\tilde{\mathcal{L}}_k(pM,\nu')$. This is easily verified by the inner product of Maass cusp forms, but we cannot take the inner product of two Eisenstein series. We will use the discussion above in this section, especially Lemma 4.16, to interpret the estimates between level $M$ and level $pM$ involving Eisenstein series in detail.

The following lines sketch the proof in [12, Section 5]. Let

$$\Phi(u):=\frac{1}{8}\sqrt{\frac{\pi}{2}}u^{-\frac{1}{2}}J_{\frac{9}{2}}(u),\quad u\ge 0.$$

where $J_s$ is the $J$-Bessel function. We have $\Phi(0)=\Phi'(0)=0$. For $s\in\mathbb{C}$, define

$$\widetilde{\Phi}(s):=\int_0^\infty J_s(u)\Phi(u)\frac{du}{u}$$

and

$$\widehat{\Phi}(r):=\frac{i|\Gamma(\frac{1+k}{2}+ir)|^2}{2\pi^2\,\mathrm{sh}\,\pi r}\left(\widetilde{\Phi}(2ir)\cos\pi(\tfrac{k}{2}+ir)-\widetilde{\Phi}(-2ir)\cos\pi\left(\tfrac{k}{2}-ir\right)\right).$$

As in [12, above (5.13)], $\widehat{\Phi}(r)>0$ for $r\in\mathbb{R}\cup i(0,\frac{1}{4}]$. At level $L=M$ or $pM$, define

$$\mathcal{L}_{\widehat{\Phi}}^{(L)}(n,n):=4\pi|n|\sum_j|\rho_j^{(L)}(n)|^2\frac{\widehat{\Phi}(r_j)}{\mathrm{ch}\,\pi r_j}$$

where the sum runs over the discrete spectrum of $\Delta_k$ on $\Gamma_0(L)$ and

$$\mathcal{M}_{\widehat{\Phi}}^{(L)}(n,n):=4\pi|n|\sum_{\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}|\rho_{\mathfrak{a}}^{(L)}(n,t)|^2\frac{\widehat{\Phi}(t)}{\mathrm{ch}\,\pi t}dt$$

where the sum runs over singular cusps of $\Gamma_0(M)$. At level $L=N$, we define $\mathcal{L}_{\widehat{\Phi}}^{(N)}$ and $\mathcal{M}_{\widehat{\Phi}}^{(N)}$ with $|n|$ changed to $|\tilde{n}|$ because $\alpha_\nu$ might be non-zero. Equation [12, before (5.14)] (also [35, Theorem 2.5] as the original reference)

$$\mathcal{L}_{\widehat{\Phi}}^{(pM)}(n,n)+\mathcal{M}_{\widehat{\Phi}}^{(pM)}(n,n)=e(-\tfrac{k}{4})\mathcal{K}_{\Phi}^{(pM)}(n,n)-\mathcal{N}_{\check{\Phi}}^{(pM)}(n,n) \tag{4.37}$$

was used to conclude

$$\mathcal{L}_{\widehat{\Phi}}^{(pM)}(n,n)\le e(-\tfrac{k}{4})\mathcal{K}_{\Phi}^{(pM)}(n,n)-\mathcal{N}_{\check{\Phi}}^{(pM)}(n,n)$$

by dropping the positive term $\mathcal{M}^{(pM)}_{\hat{\Phi}}(n,n)$ and [12, Theorem 4.3] was proved by estimating the average of the right hand side. Here we must retain this term. Recall the index $[\Gamma_0(M):\Gamma_0(pM)] \le p+1 \ll P$:

**Proposition 4.17.** *With the notations above in this subsection, for $pM \in \mathcal{Q}$ we have*

$$\mathcal{L}^{(pM)}_{\hat{\Phi}}(n,n) + \mathcal{M}^{(pM)}_{\hat{\Phi}}(n,n) \ge \frac{\mathcal{L}^{(M)}_{\hat{\Phi}}(n,n) + \mathcal{M}^{(M)}_{\hat{\Phi}}(n,n)}{[\Gamma_0(M):\Gamma_0(pM)]} \gg \frac{\mathcal{L}^{(M)}_{\hat{\Phi}}(n,n) + \mathcal{M}^{(M)}_{\hat{\Phi}}(n,n)}{P}. \tag{4.38}$$

*Proof of Proposition 4.17.* First we apply (4.35) in Lemma 4.16 with levels $M$ and $pM$. Here we take $q=1$, $\nu^{(S)} = \nu^{(T)} = (\frac{|D|}{\cdot})\nu_\theta^{2k}$ for $k = \pm\frac12$ and $I(M,pM) = [\Gamma_0(M):\Gamma_0(pM)]$ to get

$$\begin{aligned}
&\frac{1}{[\Gamma_0(M):\Gamma_0(pM)]}\Bigg(\sum_{r_j \text{ of } \Gamma_0(M)} |\rho_j^{(M)}(n)|^2 \Lambda\left(\sigma+\tfrac{it}{2}, \sigma-\tfrac{it}{2}, r_j\right) \\
&\qquad + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(M)}} \int_{-\infty}^{\infty} \frac{|\varphi^{(M)}_{\mathfrak{a}n}(\frac12+ir)|^2 \Lambda\left(\sigma+\frac{it}{2}, \sigma-\frac{it}{2}, r\right) dr}{4n\,\Gamma\left(\frac12+\frac{k}{2}-ir\right)\Gamma\left(\frac12+\frac{k}{2}-ir\right)}\Bigg) \\
&\qquad + \sum_{r_j \text{ of } \Gamma_0(pM)} |\rho_j^{\mathrm{comp}}(n)|^2 \Lambda\left(\sigma+\tfrac{it}{2}, \sigma-\tfrac{it}{2}, r_j\right) \\
&\qquad + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(pM)}} \int_{-\infty}^{\infty} \frac{|\varphi^{\mathrm{comp}}_{\mathfrak{a}n}(\frac12+ir)|^2 \Lambda\left(\sigma+\frac{it}{2}, \sigma-\frac{it}{2}, r\right) dr}{4n\,\Gamma\left(\frac12+\frac{k}{2}-ir\right)\Gamma\left(\frac12+\frac{k}{2}-ir\right)} \\
&= \sum_{r_j \text{ of } \Gamma_0(pM)} |\rho_j^{(pM)}(n)|^2 \Lambda\left(\sigma+\tfrac{it}{2}, \sigma-\tfrac{it}{2}, r_j\right) \\
&\qquad + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(pM)}} \int_{-\infty}^{\infty} \frac{|\varphi^{(pM)}_{\mathfrak{a}n}(\frac12+ir)|^2 \Lambda\left(\sigma+\frac{it}{2}, \sigma-\frac{it}{2}, r\right) dr}{4n\,\Gamma\left(\frac12+\frac{k}{2}-ir\right)\Gamma\left(\frac12+\frac{k}{2}-ir\right)}
\end{aligned} \tag{4.39}$$

Following Proskurin, we multiply a function of $t$ defined by [33, (53)] on both sides of the above formula, integrate $t$ from 0 to $\infty$, and pass to the limit $\sigma \to 1^+$. In addition we take the test function $\varphi$ in [33, (34)] to be our $\Phi$ here. What we get simplifies to (see [33, (83)])

$$\begin{aligned}
&\frac{1}{[\Gamma_0(M):\Gamma_0(pM)]}\Bigg(\mathcal{L}^{(M)}_{\hat{\Phi}}(n,n) + \mathcal{M}^{(M)}_{\hat{\Phi}}(n,n) + 4\pi n \sum_{r_j \text{ of } \Gamma_0(pM)} |b_j^{\mathrm{comp}}(n)|^2 \frac{\widehat{\Phi}(r_j)}{\operatorname{ch} \pi r_j} \\
&\qquad + n \sum_{\mathfrak{a} \text{ of } \Gamma_0(pM)} \int_{-\infty}^{\infty} |b_{\mathfrak{a}}^{\mathrm{comp}}(n,r)|^2 \frac{\widehat{\Phi}(r)}{\operatorname{ch} \pi r} dr \\
&\qquad = \mathcal{L}^{(pM)}_{\hat{\Phi}}(n,n) + \mathcal{M}^{(pM)}_{\hat{\Phi}}(n,n)
\end{aligned}$$

Our notations are consistent with $\varphi_{\mathfrak{a}\ell}$ in [33], $b_{\mathfrak{a}}(\ell, r)$ in [35], and $\widehat{\ }$ in both the references. Since [12, below (5.13)]

$$\widehat{\Phi}(r) > 0 \quad \text{for} \quad r \in \mathbb{R} \cup i(0, \tfrac14],$$

we can drop the extra terms with superscript "comp" by positivity to get the desired inequality. ☐

With Proposition 4.17, since $|\mathcal{Q}| \asymp \frac{P}{\log P}$, summing (4.37) over $\mathcal{Q}$ gives

$$\frac{1}{\log P}\left(\mathcal{L}_{\hat{\Phi}}^{(M)}(n,n)+\mathcal{M}_{\hat{\Phi}}^{(M)}(n,n)\right) \ll \sum_{pM\in\mathcal{Q}}\left|\mathcal{K}_{\Phi}^{(pM)}(n,n)\right| + \sum_{pM\in\mathcal{Q}}\left|\mathcal{N}_{\check{\Phi}}^{(pM)}(n,n)\right|. \tag{4.40}$$

When $n$ is square-free, it was shown in [12, §5.3-5.5] that the right hand side of (4.40) is bounded by $O(n^{\frac{131}{294}+\varepsilon}x^3)$, where $P = n^{\frac{1}{7}} \Rightarrow \log P \ll n^{\varepsilon}$.

Next, we will prove the bound on the right hand side of (4.40) when $n$ is not square-free. The estimates in [12, §5.3-5.4] depend on their Proposition 5.2, which is the only place that requires $n$ to be square-free. That proposition is a special case of [49, (19)], so we apply the general estimate from Waibel's paper here. For $\mu \in \{-1,0,1\}$, $n \in \mathbb{N}$ and $x \geq 1$, define

$$K_{\mu}^{(N)}(n,x) := \sum_{N|c\leq x}\frac{S(n,n,c,\nu)}{c}e\left(\frac{2\mu n}{c}\right).$$

**Proposition 4.18** ([49, (19)])**.** *Suppose that* $N \equiv 0 \pmod 8$*, that* $\mu \in \{-1,0,1\}$*, that* $n > 0$ *is factorized as* $n = tu^2w^2$ *where* $t$ *is square-free,* $u|N^{\infty}$ *and* $(w,N)=1$*, then*

$$\sum_{Q\in\mathcal{Q}}|K_{\mu}^{(Q)}(n,x)| \ll_{N,\varepsilon} \left(xP^{-\frac{1}{2}} + xun^{-\frac{1}{2}} + (x+n)^{\frac{5}{8}}\left(x^{\frac{1}{4}}P^{\frac{3}{8}} + n^{\frac{1}{8}}x^{\frac{1}{8}}P^{\frac{1}{4}}\right)\right)(nx)^{\varepsilon}.$$

Note that in the proof, Waibel chose $P$ to be $n^{\frac{1}{7}}$. By using the above proposition in each place of [12, §5.3-5.4] where [12, Proposition 5.2] was applied, we obtain new estimates that are recorded here:

[12, (5.19)] $\quad \ll \left(\ell^{-\frac{1}{2}}n^{\frac{3}{7}} + \ell^{-\frac{1}{4}}n^{\frac{23}{56}} + \ell^{-2}u\right)(\ell n)^{\varepsilon}.$

[12, (5.22)] $\quad \ll \left(\ell^{\frac{11}{6}}n^{\frac{3}{7}} + \ell^{\frac{25}{12}}n^{\frac{23}{56}} + \ell^{\frac{1}{3}}u\right)(\ell n)^{\varepsilon}.$

[12, (5.24)] $\quad \ll n^{\frac{3}{7}+\frac{5}{6}\beta+\varepsilon} + n^{\frac{23}{56}+\frac{13}{12}\beta+\varepsilon} + un^{-\frac{2}{3}\beta+\varepsilon}.$

[12, after balancing (5.26)] $\quad \ll n^{\frac{137}{294}+\varepsilon} + un^{\frac{1}{147}+\varepsilon}.$

[12, (5.28)] $\quad \displaystyle\sum_{pM\in\mathcal{Q}}\left|\mathcal{K}_{\Phi}^{(pM)}(n,n)\right| \ll n^{\frac{131}{294}+\varepsilon} + un^{-\frac{2}{147}+\varepsilon}.$

[12, (5.29)] $\quad \displaystyle\sum_{pM\in\mathcal{Q}}\left|\mathcal{N}_{\check{\Phi}}^{(pM)}(n,n)\right| \ll n^{\frac{3}{7}+\varepsilon} + un^{\varepsilon}.$

Based on the last two estimates and (4.40), we derive

$$\mathcal{L}_{\hat{\Phi}}^{(M)}(n,n)+\mathcal{M}_{\hat{\Phi}}^{(M)}(n,n) \ll (n^{\frac{131}{294}}+u)n^{\varepsilon}. \tag{4.41}$$

Finally we transfer the bound to level $N$. Apply Lemma 4.16 again with level $N$ and level $M$, where we have $\nu^{(N)} = \nu$, $\nu^{(M)} = \nu' = (\frac{|D|}{\cdot})\nu_{\theta}^{2k}$, $q = B$ and $qn_{(N)} = B\tilde{n}$. For $\ell \in \{m,n\}$, we factor $|B\tilde{\ell}| = t_{\ell}u_{\ell}^2m_{\ell}^2$ in the statement of Proposition 4.15. Here

$$\rho_j^{(N)}(n) = \rho_j^{(M)}(B\tilde{n}) \quad \text{and} \quad \rho_{\mathfrak{a}}^{(N)}(n,r) = \rho_{\mathfrak{a}}^{(M)}(B\tilde{n},r)$$

for $r_j$ a spectral parameter of $\Delta_k$ on $\Gamma_0(N)$ and $\mathfrak{a}$ a singular cusp of $(\Gamma_0(N),\nu)$. As in the proof of Proposition 4.17 above, we integrate (4.35) to a result involving $\widehat{\Phi}$, drop the extra terms as $\widehat{\Phi}(r) > 0$ for

$r \in \mathbb{R} \cup i(0, \frac{1}{4}]$, and get

$$\begin{aligned}
&|\tilde{n}| \Bigg( \sum_{r_j \text{ of } \Gamma_0(N)} \frac{|\rho_j^{(N)}(n)|^2}{\operatorname{ch} \pi r_j} \widehat{\Phi}(r_j) + \sum_{\substack{\text{singular } \mathfrak{a} \\ \text{of } \Gamma_0(N)}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{|\rho_{\mathfrak{a}}^{(N)}(n,r)|^2}{\operatorname{ch} \pi r} \widehat{\Phi}(r) dr \Bigg) \\
&= \mathcal{L}_{\widehat{\Phi}}^{(N)}(n,n) + \mathcal{M}_{\widehat{\Phi}}^{(N)}(n,n) \ll_\nu \mathcal{L}_{\widehat{\Phi}}^{(M)}(B\tilde{n}, B\tilde{n}) + \mathcal{M}_{\widehat{\Phi}}^{(M)}(B\tilde{n}, B\tilde{n}) \\
&\ll_{\nu,\varepsilon} \left( |B\tilde{n}|^{\frac{131}{294}} + u_n \right) |B\tilde{n}|^{\varepsilon} \\
&\ll_{\nu,\varepsilon} \left( |\tilde{n}|^{\frac{131}{294}} + u_n \right) |\tilde{n}|^{\varepsilon}.
\end{aligned} \tag{4.42}$$

Following from the same argument as [12, §5.5, (5.31-33)], when $x \geq 1$, $k = \pm\frac{1}{2}$ and $\tilde{n} > 0$ we have

$$\widehat{\Phi}(r)^{-1} \ll x^{3-k} \quad \text{for } |r| \leq x$$

and get Proposition 4.15. When $n < 0$ it follows from the relationship (1.14) and (2.17).

## 4.5 Proof of Theorem 1.7, mixed-sign case

In this section we prove Theorem 1.14 in the case $\tilde{m}\tilde{n} < 0$. For simplicity let

$$A(m,n) := \left( \tilde{m}^{\frac{131}{294}} + u_m \right)^{\frac{1}{2}} \left( |\tilde{n}|^{\frac{131}{294}} + u_n \right)^{\frac{1}{2}} \ll |\tilde{m}\tilde{n}|^{\frac{131}{588}} + \tilde{m}^{\frac{131}{588}} u_n^{\frac{1}{2}} + |\tilde{n}|^{\frac{131}{588}} u_m^{\frac{1}{2}} + (u_m u_n)^{\frac{1}{2}}$$

and

$$\begin{aligned}
A_u(m,n) &:= A(m,n)^{\frac{1}{4}} |\tilde{m}\tilde{n}|^{\frac{3}{16}} \\
&\ll |\tilde{m}\tilde{n}|^{\frac{143}{588}} + \tilde{m}^{\frac{143}{588}} |\tilde{n}|^{\frac{3}{16}} u_n^{\frac{1}{8}} + \tilde{m}^{\frac{3}{16}} u_m^{\frac{1}{8}} |\tilde{n}|^{\frac{143}{588}} + |\tilde{m}\tilde{n}|^{\frac{3}{16}} (u_m u_n)^{\frac{1}{8}}.
\end{aligned} \tag{4.43}$$

Moreover, all implicit constants for bounds in this section depend on $\nu$ and $\varepsilon$ and we drop the subscripts unless specified. Recall the notations in Settings 4.8 and 4.9. For the exceptional spectrum $r_j \in i(0, \frac{1}{4}]$ of the Laplacian $\Delta_k$ on $\Gamma = \Gamma_0(N)$, we have $2 \operatorname{Im} r_\Delta \leq \theta$ assuming $H_\theta$ (2.15) by Proposition 4.7 and $\operatorname{Im} r_j$ has a positive lower bound $\underline{t} > 0$ depending on $N$.

**Proposition 4.19.** *With the same setting as Theorem 1.7, when $2x \geq A_u(m,n)^2$, we have*

$$\begin{aligned}
&\sum_{\substack{x < c \leq 2x \\ N | c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j \in i(0, \frac{1}{4}]} (2^{2s_j - 1} - 1) \tau_j(m,n) \frac{x^{2s_j - 1}}{2s_j - 1} \\
&\qquad \ll \left( x^{\frac{1}{6}} + A_u(m,n) \right) |\tilde{m}\tilde{n}x|^{\varepsilon}.
\end{aligned} \tag{4.44}$$

We first prove that Proposition 4.19 implies Theorem 1.7. For each $j$, let $\rho_j(n)$ denote the coefficients of an orthonormal basis $\{v_j(\cdot)\}$ of $\tilde{\mathcal{L}}_k(N, \nu)$. For each singular cusp $\mathfrak{a}$ of $\Gamma = \Gamma_0(N)$, let $E_{\mathfrak{a}}(\cdot, s)$ be the associated Eisenstein series and $\rho_{\mathfrak{a}}(n,r)$ be defined as in (4.3).

Recall the definition of $\tau_j(m,n)$ in Theorem 1.7 and $2 \operatorname{Im} r_j = 2s_j - 1 \in (0, \frac{1}{2}]$ and $\underline{t} > 0$ as the lower bound of $\operatorname{Im} r_j$ depending on $\nu$. The sum to be estimated is

$$\sum_{N|c\le X} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} \tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}. \tag{4.45}$$

For $r_j \in i(0,\frac14]$, by Proposition 4.15 we have

$$\frac{\tau_j(m,n)}{2s_j-1} \ll |\rho_j(m)\rho_j(n)||\tilde{m}\tilde{n}|^{1-s_j} \ll A(m,n)|\tilde{m}\tilde{n}|^{\frac12-s_j+\varepsilon}. \tag{4.46}$$

When $X \ll A_u(m,n)^2$, since $A(m,n)\le 2|\tilde{m}\tilde{n}|^{\frac14}$,

$$\begin{aligned}\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1} &\ll A(m,n)|\tilde{m}\tilde{n}|^{\frac12-s_j+\varepsilon}A_u(m,n)^{4s_j-2}\\ &= A(m,n)^{s_j+\frac12}|\tilde{m}\tilde{n}|^{\frac18-\frac14 s_j} \ll A_u(m,n).\end{aligned} \tag{4.47}$$

So in this case we get Theorem 1.7 where the $\tau_j$ terms are absorbed in the errors.

When $X\ge A_u(m,n)^2$, the segment for summing Kloosterman sums on $1\le c\le A_u(m,n)^2$ contributes a $O_{\nu,\varepsilon}(A_u(m,n)|\tilde{m}\tilde{n}|^{\varepsilon})$ by condition (2) of Definition 1.6. The segment for $A_u(m,n)^2\le c\le X$ can be broken into no more than $O(\log X)$ dyadic intervals $x<c\le 2x$ with $A_u(m,n)^2\le x\le \frac{X}{2}$ and we use Proposition 4.19 for both the Kloosterman sum and the $\tau_j$ terms. In summing dyadic intervals, for each $r_j\in i(0,\frac14]$, we get

$$\begin{aligned}&\sum_{\ell=1}^{\lceil \log_2(X/A_u(m,n)^2)\rceil} \frac{(2^{2s_j-1}-1)\tau_j(m,n)}{2s_j-1}\left(\frac{X}{2^\ell}\right)^{2s_j-1}\\ &\qquad = \frac{\tau_j(m,n)}{2s_j-1}X^{2s_j-1}\left(1-2^{(1-2s_j)\lceil \log_2(X/A_u(m,n)^2)\rceil}\right).\end{aligned}$$

The difference between the above quantity and $\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}$ in (4.45) is

$$\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}\cdot 2^{(1-2s_j)\lceil \log_2(X/A_u(m,n)^2)\rceil} \ll \frac{\tau_j(m,n)}{2s_j-1}A_u(m,n)^{4s_j-2} \ll A_u(m,n). \tag{4.48}$$

by (4.46). In conclusion, for $X\ge A_u(m,n)^2$ we get

$$\begin{aligned}&\sum_{N|c\le X} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} \tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}\\ &= \sum_{A_u(m,n)^2<c\le X} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} \tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1} + O(A_u(m,n)|\tilde{m}\tilde{n}|^{\varepsilon})\\ &= \sum_{\ell=1}^{\lceil \log_2(X/A_u(m,n)^2)\rceil}\left(\sum_{\frac{X}{2^\ell}<c\le\frac{X}{2^{\ell-1}}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} \frac{(2^{2s_j-1}-1)\tau_j(m,n)}{2s_j-1}\left(\frac{X}{2^\ell}\right)^{2s_j-1}\right)\\ &\quad + O(A_u(m,n)|\tilde{m}\tilde{n}|^{\varepsilon})\\ &\ll \left(X^{\frac16}+A_u(m,n)\right)|\tilde{m}\tilde{n}X|^{\varepsilon}\end{aligned}$$

where the second equality follows from (4.48) and the last inequality is by Proposition 4.19.

It remains to prove Proposition 4.19. For $r_j \in i(0, \frac{1}{4}]$, by Proposition 4.15 we have

$$\sqrt{|\tilde{m}\tilde{n}|}\; \overline{\rho_j(m)}\rho_j(n) \ll A(m,n)|\tilde{m}\tilde{n}|^{\varepsilon}.$$

Applying Lemma 4.13 where $2t_j = 2\operatorname{Im} r_j = 2s_j - 1$, recalling the definition of $\tau_j$ in Theorem 1.7 and $a = 4\pi\sqrt{|\tilde{m}\tilde{n}|}$ in Setting 4.8, we get

$$\begin{aligned} &2i^k \cdot 4\sqrt{|\tilde{m}\tilde{n}|}\; \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j) \\ &\quad = (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} + O\left(A(m,n)|\tilde{m}\tilde{n}|^{\varepsilon}\left(|\tilde{m}\tilde{n}|^{-t_j}x^{2t_j-\delta}+1\right)\right). \end{aligned} \tag{4.49}$$

The error term is $O(A(m,n)|\tilde{m}\tilde{n}|^{\varepsilon})$ when $2t_j \le \delta$ and is $O(x^{\frac{1}{2}-\delta}|\tilde{m}\tilde{n}|^{\varepsilon})$ when $t_j = \frac{1}{4}$. Thanks to Proposition 4.7 we can choose $\delta > \theta$ ($\delta = \frac{1}{3} > \frac{7}{64}$ in the end) and $t_j < \frac{1}{4}$ implies $2t_j \le \theta < \delta$. With the help of (4.49) we break up the left hand side of (4.44) as

$$\begin{aligned} &\left|\sum_{\substack{x<c\le 2x\\ N|c}}\frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac{1}{4}]}(2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}\right| \\ \le &\left|\sum_{\substack{x<c\le 2x\\ N|c}}\frac{S(m,n,c,\nu)}{c} - \sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right)\right| + O\left(\left(x^{\frac{1}{2}-\delta}+A(m,n)\right)|\tilde{m}\tilde{n}|^{\varepsilon}\right) \\ &+\left|\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right) - 8i^k\sqrt{|\tilde{m}\tilde{n}|}\sum_{r_j\in i(0,\frac{1}{4}]}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \\ =: &\; S_1 + O\left(\left(x^{\frac{1}{2}-\delta}+A(m,n)\right)|\tilde{m}\tilde{n}|^{\varepsilon}\right) + S_2. \end{aligned} \tag{4.50}$$

Recall $T \asymp x^{1-\delta}$. The first sum $S_1$ above can be estimated by condition (2) of Definition 1.6 as

$$S_1 \le \sum_{\substack{x-T\le c\le x\\ 2x\le c\le 2x+2T\\ N|c}}\frac{|S(m,n,c,\nu)|}{c} \ll_{\delta,\varepsilon} x^{\frac{1}{2}-\delta}|\tilde{m}\tilde{n}x|^{\varepsilon} \tag{4.51}$$

We then prove a bound for $S_2$. Following from the trace formula (4.8),

$$S_2 = 8\sqrt{|\tilde{m}\tilde{n}|}\left|\sum_{r_j\ge 0}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j) + \sum_{\text{singular }\mathfrak{a}}\frac{1}{4\pi}\int_{-\infty}^{\infty}\overline{\rho_{\mathfrak{a}}(m,r)}\rho_{\mathfrak{a}}(n,r)\frac{\check{\phi}(r)}{\operatorname{ch}\pi r}dr\right|.$$

When estimating $S_2$, we focus on the discrete spectrum $r_j \ge 0$, because the bounds provided by Proposition 4.14 and Proposition 4.15 for $r_j \in I$ for any interval $I$ are the same as those provided for for $|r| \in I$ in the continuous spectrum. For $r \in [0,1)$, we apply Proposition 4.15, (4.14) and (4.15) to get

$$\sqrt{\tilde{m}|\tilde{n}|}\sum_{r\in[0,1)}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \ll A(m,n)|\tilde{m}\tilde{n}|^{\varepsilon}. \tag{4.52}$$

For $r \in [1, \frac{a}{8x})$, we apply Proposition 4.15 and (4.16) with $\check{\phi}(r) \ll e^{-\frac{r}{2}}$. Since

$$S(R) := \sqrt{\tilde{m}|\tilde{n}|} \sum_{r \in [1,R]} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \right| \ll A(m,n) R^3 |\tilde{m}\tilde{n}|^{\varepsilon} \tag{4.53}$$

by Cauchy-Schwarz, we have

$$\begin{aligned} \sqrt{\tilde{m}|\tilde{n}|} \sum_{r \in [1, \frac{a}{8x})} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| &\ll \sqrt{\tilde{m}|\tilde{n}|} \sum_{r \in [1, \frac{a}{8x})} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \right| e^{-\frac{r_j}{2}} \\ &\ll e^{-\frac{r}{2}} S(r) \Big|_{r=1}^{\frac{a}{8x}} + \int_1^{\frac{a}{8x}} S(r) e^{-\frac{r}{2}} dr \\ &\ll A(m,n) |\tilde{m}\tilde{n}x|^{\varepsilon} \left( 1 + \int_1^{\frac{a}{8x}} e^{-\frac{r}{2}} r^3 dr \right) \\ &\ll A(m,n) |\tilde{m}\tilde{n}x|^{\varepsilon}. \end{aligned} \tag{4.54}$$

For $r \in [\frac{a}{8x}, \frac{a}{x})$, we apply Proposition 4.14 on $\tilde{m}$, Proposition 4.15 on $\tilde{n}$ and (4.16) with $\check{\phi}(r) \ll \frac{1}{r} \ll \frac{x}{a}$ to get

$$\begin{aligned} &\sqrt{\tilde{m}|\tilde{n}|} \sum_{\frac{a}{8x} \le r < \frac{a}{x}} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| \\ &\ll \left( \frac{a}{x} + \tilde{m}^{\frac{1}{4}} \right) \left( \frac{a}{x} \right)^{\frac{1}{2}} \left( |\tilde{n}|^{\frac{131}{294}} + u_n \right)^{\frac{1}{2}} |\tilde{m}\tilde{n}x|^{\varepsilon} \\ &\ll \left( A(m,n) \left( \frac{a}{x} \right)^{\frac{3}{2}} + \tilde{m}^{\frac{1}{4}} \left( |\tilde{n}|^{\frac{131}{294}} + u_n \right)^{\frac{1}{2}} \left( \frac{a}{x} \right)^{\frac{1}{2}} \right) |\tilde{m}\tilde{n}x|^{\varepsilon}. \end{aligned} \tag{4.55}$$

Exchanging the propositions applied on $\tilde{m}$ and $\tilde{n}$ gives a symmetric estimate. These two estimates conclude

$$\begin{aligned} &\sqrt{\tilde{m}|\tilde{n}|} \sum_{\frac{a}{8x} \le r < \frac{a}{x}} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \check{\phi}(r_j) \right| \\ &\ll \left( \frac{a}{x} \right)^{\frac{1}{2}} \left\{ A(m,n) \frac{a}{x} + \min \left( \tilde{m}^{\frac{1}{4}} \left( |\tilde{n}|^{\frac{131}{294}} + u_n \right)^{\frac{1}{2}}, |\tilde{n}|^{\frac{1}{4}} \left( \tilde{m}^{\frac{131}{294}} + u_m \right)^{\frac{1}{2}} \right) \right\} |\tilde{m}\tilde{n}x|^{\varepsilon} \\ &\ll \left( A(m,n) \left( \frac{a}{x} \right)^{\frac{3}{2}} + |\tilde{m}\tilde{n}|^{\frac{1}{8}} A(m,n)^{\frac{1}{2}} \left( \frac{a}{x} \right)^{\frac{1}{2}} \right) |\tilde{m}\tilde{n}x|^{\varepsilon}. \end{aligned} \tag{4.56}$$

where in the last inequality we applied $\min(B,C) \le \sqrt{BC}$ and the definition of $A(m,n)$ at the beginning of this subsection.

Let

$$P(m,n) := 2|\tilde{m}\tilde{n}|^{\frac{1}{8}} A(m,n)^{-\frac{1}{2}} \ge 1.$$

Divide $r \ge \max(\frac{a}{x}, 1)$ into two parts: $\max\left(\frac{a}{x}, 1\right) \le r < P(m,n)$ and $r \ge \max\left(\frac{a}{x}, 1, P(m,n)\right)$. We apply

Proposition 4.15 on the first range and (4.16) with $\check{\phi}(r) \ll r^{-\frac{3}{2}}$ to get

$$\begin{aligned}
\sqrt{\tilde{m}|\tilde{n}|} &\sum_{\max(\frac{a}{x},1)\le r_j<P(m,n)} \left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \\
&\ll \sqrt{\tilde{m}|\tilde{n}|} \sum_{\max(\frac{a}{x},1)\le r_j<P(m,n)} \left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\right| r^{-\frac{3}{2}} \\
&\ll r^{-\frac{3}{2}} S(r)\Big|_{r=\max(\frac{a}{x},1)}^{P(m,n)} + \int_{\max(\frac{a}{x},1)}^{P(m,n)} r^{-\frac{5}{2}} S(r)dr \\
&\ll |\tilde{m}\tilde{n}|^{\frac{3}{16}} A(m,n)^{\frac{1}{4}} |\tilde{m}\tilde{n}x|^{\varepsilon}
\end{aligned} \tag{4.57}$$

by partial summation. We divide the second range into dyadic intervals $C \le r_j < 2C$ and apply Proposition 4.14 and (4.16) with $\check{\phi}(r) \ll \min(r^{-\frac{3}{2}}, r^{-\frac{5}{2}}\frac{x}{T})$ to get

$$\begin{aligned}
\sqrt{\tilde{m}|\tilde{n}|} &\sum_{C\le r_j<2C} \left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \\
&\ll \min\left(C^{-\frac{3}{2}}, C^{-\frac{5}{2}}\frac{x}{T}\right)\left(C^2 + (\tilde{m}^{\frac{1}{4}} + |\tilde{n}|^{\frac{1}{4}})C + |\tilde{m}\tilde{n}|^{\frac{1}{4}}\right)|\tilde{m}\tilde{n}x|^{\varepsilon} \\
&\ll \left(\min\left(C^{\frac{1}{2}}, C^{-\frac{1}{2}}\frac{x}{T}\right) + (\tilde{m}^{\frac{1}{4}} + |\tilde{n}|^{\frac{1}{4}})C^{-\frac{1}{2}} + |\tilde{m}\tilde{n}|^{\frac{1}{4}}C^{-\frac{3}{2}}\right)|\tilde{m}\tilde{n}x|^{\varepsilon}.
\end{aligned} \tag{4.58}$$

Next we sum over dyadic intervals. For the first term $\min(C^{\frac{1}{2}}, C^{-\frac{1}{2}}\frac{x}{T})$, when

$$\min\left(C^{\frac{1}{2}}, C^{-\frac{1}{2}}\frac{x}{T}\right) = C^{\frac{1}{2}}: \qquad \sum_{\substack{j\ge1:\ 2^jC=\frac{x}{T}\\ C\ge P(m,n)}} C^{\frac{1}{2}} \le \sum_{j=1}^{\infty} 2^{-\frac{j}{2}}\left(\frac{x}{T}\right)^{\frac{1}{2}} \ll \left(\frac{x}{T}\right)^{\frac{1}{2}},$$

and when

$$\min\left(C^{\frac{1}{2}}, C^{-\frac{1}{2}}\frac{x}{T}\right) = C^{-\frac{1}{2}}\frac{x}{T}: \qquad \sum_{j\ge0:\ C=2^j\frac{x}{T}} C^{-\frac{1}{2}}\frac{x}{T} \le \sum_{j=0}^{\infty} 2^{-\frac{j}{2}}\left(\frac{x}{T}\right)^{\frac{1}{2}} \ll \left(\frac{x}{T}\right)^{\frac{1}{2}}.$$

So after summing up from (4.58) and recalling $T \asymp x^{1-\delta}$ in Setting 4.8, we have

$$\begin{aligned}
\sqrt{\tilde{m}|\tilde{n}|} &\sum_{r_j\ge\max(\frac{a}{x},1,P(m,n))} \left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \\
&\ll \left(\left(\frac{x}{T}\right)^{\frac{1}{2}} + (\tilde{m} + |\tilde{n}|)^{\frac{1}{4}}|\tilde{m}\tilde{n}|^{-\frac{1}{16}}A(m,n)^{\frac{1}{4}} + |\tilde{m}\tilde{n}|^{\frac{1}{16}}A(m,n)^{\frac{3}{4}}\right)|\tilde{m}\tilde{n}|^{\varepsilon} \\
&\ll \left(x^{\frac{\delta}{2}} + |\tilde{m}\tilde{n}|^{\frac{3}{16}}A(m,n)^{\frac{1}{4}}\right)|\tilde{m}\tilde{n}x|^{\varepsilon},
\end{aligned} \tag{4.59}$$

where the last inequality is by $|\tilde{m}\tilde{n}|^{\frac{1}{4}} \gg A(m,n)$. Combining (4.57) and (4.59) we have

$$\sqrt{\tilde{m}|\tilde{n}|} \sum_{r\ge\max(\frac{a}{x},1)} \left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\check{\phi}(r_j)\right| \ll \left(x^{\frac{\delta}{2}} + A_u(m,n)\right)|\tilde{m}\tilde{n}x|^{\varepsilon}. \tag{4.60}$$

*Proof of Proposition 4.19.* Clearly $A_u(m,n) \ge A(m,n)$. Combining (4.50), (4.51), (4.52), (4.54), (4.56), and

(4.60) we get

$$\sum_{\substack{x<c\leq 2x \\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac{1}{4}]} (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}$$
$$\ll \left(x^{\frac{1}{2}-\delta} + A_u(m,n) + A(m,n)\left(\frac{a}{x}\right)^{\frac{3}{2}} + |\tilde{m}\tilde{n}|^{\frac{1}{8}} A(m,n)^{\frac{1}{2}}\left(\frac{a}{x}\right)^{\frac{1}{2}} + x^{\frac{\delta}{2}}\right)|\tilde{m}\tilde{n}x|^{\varepsilon}. \tag{4.61}$$

Since $2x \geq A_u(m,n)^2$ by assumption, we have

$$\frac{a}{x} \ll |\tilde{m}\tilde{n}|^{\frac{1}{8}} A(m,n)^{-\frac{1}{2}},$$

which implies both

$$A(m,n)\left(\frac{a}{x}\right)^{\frac{3}{2}} \ll A_u(m,n) \quad \text{and} \quad |\tilde{m}\tilde{n}|^{\frac{1}{8}} A(m,n)^{\frac{1}{2}}\left(\frac{a}{x}\right)^{\frac{1}{2}} \ll A_u(m,n). \tag{4.62}$$

Taking $\delta = \frac{1}{3}$ we get the desired bound.

$\square$

# Chapter 5

# Sums of Kloosterman sums: uniform bounds, same-sign case

In this chapter we prove the complementary case $\tilde{m}\tilde{n} > 0$ in Theorem 1.7. The difference of the proof in this chapter from the previous one is due to the difference between Theorem 5.1 and Theorem 4.10.

## 5.1 Kuznetsov trace formula in the same-sign case

Let $k \in \mathbb{Z} + \frac{1}{2}$, $N$ be a positive integer, and $\mathfrak{a}$ be a singular cusp for the weight $k$ multiplier system $\nu$ on $\Gamma = \Gamma_0(N)$. Recall the definition of the Eisenstein series associated to $\mathfrak{a}$ in (4.1). For $m > 0$, recall the definition of Poincaré series (4.2). Recall the Fourier expansion of the Poincaré series in (4.5) and of the Eisenstein series in (4.3).

Suppose $m$ and $n$ are positive integers and recall the definition of $\alpha_\infty$ in 1.11. Recall Setting 4.8 and Setting 4.9. In this chapter, we need the following transformations of $\phi$:

$$\widetilde{\phi}(r) = \int_0^\infty J_{r-1}(u)\phi(u)\frac{du}{u} \tag{5.1}$$

and for $k \geq 0$,

$$\widehat{\phi}(r) := \pi^2 e^{\frac{(1+k)\pi i}{2}} \frac{\int_0^\infty \left(\cos(\frac{k\pi}{2} + \pi i r)J_{2ir}(u) - \cos(\frac{k\pi}{2} - \pi i r)J_{-2ir}(u)\right)\phi(u)\frac{du}{u}}{\operatorname{sh}(\pi r)(\operatorname{ch}(2\pi r) + \cos \pi k)\Gamma(\frac{1}{2} - \frac{k}{2} + ir)\Gamma(\frac{1}{2} - \frac{k}{2} - ir)} \tag{5.2}$$

with the corrected version of [42, (2.12)]

$$\widehat{\phi}\left(\tfrac{i}{4}\right) = \begin{cases} e^{\frac{\pi i}{4}} \int_0^\infty \cos(u)\phi(u)u^{-\frac{3}{2}}du & k = \frac{1}{2}, \\ \frac{1}{2}e^{\frac{3\pi i}{4}} \int_0^\infty \sin(u)\phi(u)u^{-\frac{3}{2}}du & k = \frac{3}{2}. \end{cases} \tag{5.3}$$

For an integer $l \geq 1$, let $B_l$ denote an orthonormal basis for the space of holomorphic cusp forms $S_{k+2l}(N,\nu)$ and

$$\mathscr{B}_k := \bigcup_{l=1}^{\infty} B_l.$$

Suppose that the Fourier expansion of each $F \in \mathscr{B}_k$ is given by

$$F(z) := \sum_{n=1}^{\infty} a_F(n) e(\tilde{n} z). \tag{5.4}$$

Let $w_F$ denote the weight of $F \in \mathscr{B}_k$. Here is the trace formula:

**Theorem 5.1** ([33, §6]). *Suppose $\nu$ is a multiplier system of weight $k = \frac{1}{2}$ or $\frac{3}{2}$ on $\Gamma$. Let $\{v_j(\cdot)\}$ be an orthonormal basis of $\tilde{\mathcal{L}}_k(N, \nu)$ and $E_{\mathfrak{a}}(\cdot, s)$ be the Eisenstein series associated to a singular cusp $\mathfrak{a}$. Let $\rho_j(n)$ denote the $n$-th Fourier coefficient of $v_j$. Let $\varphi_{\mathfrak{a}n}(\frac{1}{2} + ir)$ or $\rho_{\mathfrak{a}}(n, r)$ denote the $n$-th Fourier coefficient of $E_{\mathfrak{a}}(\cdot, \frac{1}{2} + ir)$ as in (4.3). Let $\mathscr{B}_k$ and $a_F(n)$ be defined as in (5.4). Then for $\tilde{m} > 0$ and $\tilde{n} > 0$ we have*

$$\sum_{c>0} \frac{S(m, n, c, \nu)}{c} \phi\left(\frac{4\pi\sqrt{\tilde{m}\tilde{n}}}{c}\right) = \mathcal{U}_k + \mathcal{W} + \sum_{\text{singular } \mathfrak{a}} \mathcal{E}_{\mathfrak{a}}, \tag{5.5}$$

*where*

$$\mathcal{U}_k = \sum_{F \in \mathscr{B}_k} \frac{4\Gamma(w_F) e^{\pi i w_F/2}}{(4\pi)^{w_F} (\tilde{m}\tilde{n})^{(w_F - 1)/2}} \overline{a_F(m)} a_F(n) \widetilde{\phi}(w_F),$$

$$\mathcal{W} = 4\sqrt{\tilde{m}\tilde{n}} \sum_{r_j} \frac{\overline{\rho_j(m)} \rho_j(n)}{\operatorname{ch} \pi r_j} \widehat{\phi}(r_j),$$

*and*

$$\begin{aligned}
\mathcal{E}_{\mathfrak{a}} &= \int_{-\infty}^{\infty} \left(\frac{\tilde{m}}{\tilde{n}}\right)^{-ir} \frac{\overline{\varphi_{\mathfrak{a}m}\left(\frac{1}{2} + ir\right)} \varphi_{\mathfrak{a}n}\left(\frac{1}{2} + ir\right) \widehat{\phi}(r) dr}{\operatorname{ch} \pi r \, |\Gamma(\frac{1}{2} + \frac{k}{2} + ir)|^2} \\
&= 4\sqrt{\tilde{m}\tilde{n}} \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} \overline{\rho_{\mathfrak{a}}(m, r)} \rho_{\mathfrak{a}}(n, r) \frac{\widehat{\phi}(r)}{\operatorname{ch} \pi r} dr.
\end{aligned}$$

*Remark.* We clarify two points in the theorem.

(1) In the term $\mathcal{U}_k$ corresponding to holomorphic cusp forms, each function $F \in \mathscr{B}_k$ has weight $w_F = k + 2l \geq \frac{5}{2}$.

(2) The equality of the two expressions in $\mathcal{E}_{\mathfrak{a}}$ is by (4.3):

$$\sqrt{\frac{\tilde{n}}{\pi}} \rho_{\mathfrak{a}}(n, r) = \frac{e^{-\frac{\pi i k}{2}} \pi^{ir} \tilde{n}^{ir}}{\Gamma\left(\frac{1}{2} + ir + \frac{k}{2} \operatorname{sgn} \tilde{n}\right)} \varphi_{\mathfrak{a}n}\left(\frac{1}{2} + ir\right).$$

### 5.1.1 Properties of admissible multipliers

Suppose $\nu$ is a weight $k$ admissible multiplier system on $\Gamma = \Gamma_0(N)$ (Definition 1.6) with parameters $B$, $M$ and $D$. Besides Proposition 4.7, we also have:

**Proposition 5.2.** *Suppose that $\nu$ satisfies condition (1) of Definition 1.6 with $\nu' = (\frac{|D|}{\cdot})\nu_\theta^{2k}$. For $l \in \mathbb{Z}$, let $K = k + 2l \geq \frac{5}{2}$. Suppose $\{F_{j,l}(\cdot)\}_j$ is an orthonormal basis of $S_K(N, \nu)$ and $\{G_{j,l}(\cdot)\}_j$ is an orthonormal basis of $S_K(M, \nu')$. Denote $a_{F,j,l}(n)$ as the Fourier coefficient of $F_{j,l}$ and $a_{G,j,l}(n)$ as the Fourier coefficient of $G_{j,l}$. Then we have*

$$\sum_{j=1}^{\dim S_K(N,\nu)} |a_{F,j,l}(n)|^2 \ll_{N,\nu} \sum_{j=1}^{\dim S_K(M,\nu')} |a_{G,j,l}(B\tilde{n})|^2. \tag{5.6}$$

*Proof.* By condition (1) of Definition 1.6, we know that

$$\left\{[\Gamma_0(N):\Gamma_0(M)]^{-\frac{1}{2}}F_{j,l}(Bz):\ 1\le j\le \dim S_K(N,\nu)\right\} \tag{5.7}$$

is an orthonormal subset of $S_K(M,\nu')$. Since the left hand side of (2.6) is independent from the choice of basis, we expand (5.7) to an orthonormal basis of $S_K(M,\nu')$ and get the result. □

Now we start to prove a bound for the right hand side of (5.6). First we have

**Proposition 5.3** ([49, Theorem 1])**.** *For $K\in\mathbb{Z}+\frac{1}{2}$, $K\ge\frac{5}{2}$ and a quadratic character $\chi$ modulo $M$, suppose*

$$\left\{\Phi_j=\sum_{n=1}^{\infty}a_j(n)e(nz):\ 1\le j\le d:=\dim S_K(M,\chi\nu_\theta^{2K})\right\}$$

*is an orthonormal basis of $S_K(M,\chi\nu_\theta^{2K})$. For $n\ge1$, write $n=tu^2w^2$ with $t$ square-free, $u|M^\infty$ and $(w,M)=1$. Then we have*

$$\frac{\Gamma(K-1)}{(4\pi n)^{K-1}}\sum_{j=1}^{d}|a_j(n)|^2\ll_{K,M,\varepsilon}\left(n^{\frac{3}{7}}+u\right)n^{\varepsilon}.$$

Note that the implied constant in the bound above depends on $K$ when expressing Bessel functions (see [49, after (8)] and [50, Theorem 1 and p. 400]). For our proof, it is essential that the bound remains uniform across the weights. We modify the estimate and get the following proposition.

**Proposition 5.4.** *With the same setting as Proposition 5.3,*

$$\frac{\Gamma(K-1)}{(4\pi n)^{K-1}}\sum_{j=1}^{d}|a_j(n)|^2\ll_{M,\varepsilon}(n^{\frac{19}{42}}+u)n^{\varepsilon}.$$

*Proof.* We do the same preparation as [49, around (8)] to estimate the right hand side of (2.6). Let $P>1+(\log 2nM)^2$ (finally chosen to be $\asymp_M n^{\frac{1}{7}}$) and define the set of prime numbers

$$\mathcal{P}:=\ \{p\text{ prime}: P<p\le 2P,\ p\nmid 2nM\}.$$

Here we have $\#\mathcal{P}\asymp P/\log P$.

For $\{\Phi_j\}_j$ a orthonormal basis of $S_K(M,\chi\nu_\theta^{2K})$, the set $\{[\Gamma_0(M):\Gamma_0(pM)]^{-\frac{1}{2}}\Phi_j\}_j$ is an orthonormal subset of $S_K(pM,\chi\nu_\theta^{2K})$. Recall (2.6) and we have

$$\frac{\Gamma(K-1)}{(4\pi n)^{K-1}}\sum_{j=1}^{d}\frac{|a_j(n)|^2}{[\Gamma_0(M):\Gamma_0(pM)]}\le 1+2\pi i^{-K}\sum_{pM|c}\frac{S(n,n,c,\chi\nu_\theta^{2K})}{c}J_{K-1}\left(\frac{4\pi n}{c}\right). \tag{5.8}$$

For those $p\in\mathcal{P}$, $[\Gamma_0(M):\Gamma_0(pM)]\le p+1$. Summing (5.8) on $p\in\mathcal{P}$ and dividing $\#\mathcal{P}$ we get

$$\frac{\Gamma(K-1)}{(4\pi n)^{K-1}}\sum_{j=1}^{d}|a_j(n)|^2\ll P+(\log P)\sum_{p\in\mathcal{P}}\left|\sum_{pM|c}\frac{S(n,n,c,\chi\nu_\theta^{2K})}{c}J_{K-1}\left(\frac{4\pi n}{c}\right)\right|. \tag{5.9}$$

The average estimate of

$$K_{pM}^{(\mu)}(x) := \sum_{pM|c\le x} \frac{S(n,n,c,\chi\nu_\theta^{2K})}{\sqrt{c}} e\left(\frac{2\mu n}{c}\right), \quad \mu\in\{-1,0,1\}$$

can be found in [49, (19)] that for $\mu\in\{-1,0,1\}$,

$$\sum_{p\in\mathcal{P}} |K_{pM}^{(\mu)}(x)| \ll_{M,\varepsilon} \left(xun^{-\frac12} + xP^{-\frac12} + (x+n)^{\frac58}\left(x^{\frac14}P^{\frac38} + n^{\frac18}x^{\frac18}P^{\frac14}\right)\right)(nx)^{\varepsilon}. \tag{5.10}$$

We break the sum on $c\equiv 0 \pmod{pM}$ at the right hand side of (5.9) into $c\le n$ and $c\ge n$ to estimate. The uniform bound of $J$-Bessel functions is given by [51]

$$|J_\beta(x)| \le c_0 x^{-\frac13} \quad \text{for all } \beta>0 \text{ and } x>0, \tag{5.11}$$

where $c_0 = 0.7857\cdots$.

When $c\le n$, using (5.11) and [31, (10.6.1)]

$$2J_{\beta-1}'(x) = J_{\beta-2}(x) - J_\beta(x),$$

we find that

$$\left(x^{-\frac12} J_{K-1}\left(\frac{4\pi n}{x}\right)\right)' \ll n^{-\frac13}x^{-\frac76} + n^{\frac23}x^{-\frac{13}{6}}. \tag{5.12}$$

Then a partial summation using (5.10), (5.12) and (5.11) yields

$$\sum_{p\in\mathcal{P}}\left|\sum_{pM|c\le n} \frac{S(n,n,c,\chi\nu_\theta^{2K})}{c} J_{K-1}\left(\frac{4\pi n}{c}\right)\right| \ll_{M,\varepsilon} (n^{\frac{19}{42}}+u)n^{\varepsilon}. \tag{5.13}$$

When $c\ge n$, we get another bound

$$\left(x^{-\frac12} J_{K-1}\left(\frac{4\pi n}{x}\right)\right)' \ll nx^{-\frac52} \quad \text{for } x\ge n \tag{5.14}$$

by $|J_{\beta-1}(x)| \le \frac{(x/2)^{\beta-1}}{\Gamma(\beta)}$ [31, (10.14.4)] and $|J_\beta(x)|\le 1$ [31, (10.14.1)]. Remember $K\ge\frac52$ here. We do a partial summation again using (5.10) and (5.14) and get

$$\sum_{p\in\mathcal{P}}\left|\sum_{pM|c\ge n} \frac{S(n,n,c,\chi\nu_\theta^{2K})}{c} J_{K-1}\left(\frac{4\pi n}{c}\right)\right| \ll_{M,\varepsilon} (n^{\frac37}+u)n^{\varepsilon}. \tag{5.15}$$

From (5.13), (5.15), (5.9) and $P\asymp_M n^{\frac17}$, we finish the proof. □

Combining Proposition 5.2 and Proposition 5.4, one observes the following bound:

**Proposition 5.5.** *With the same setting as Proposition 5.2, we factor $B\tilde{n} = t_n u_n^2 w_n^2$ with $t_n$ square-free,*

$u_n|M^\infty$ *and* $(w_n, M) = 1$. *Then*

$$\frac{\Gamma(K-1)}{(4\pi\tilde{n})^{K-1}} \sum_{j=1}^{\dim S_K(N,\nu)} |a_{F,j,l}(n)|^2 \ll_{N,\nu,\varepsilon} (\tilde{n}^{\frac{19}{42}} + u_n)\tilde{n}^{\varepsilon}.$$

## 5.2 Bounds on $\widetilde{\phi}$ and $\widehat{\phi}$

In this section, all of the implied constants among the estimates for $\widetilde{\phi}$ and $\widehat{\phi}$ depend on $N$ and the multiplier system $\nu$ unless specified. Recall the definitions (5.1) and (5.2). To deal with the $\Gamma$-function in the denominator of $\widehat{\phi}$, we need [31, (5.6.6-7)]

$$\frac{\Gamma(x)^2}{\operatorname{ch}(\pi r)} \le |\Gamma(x+ir)|^2 \le \Gamma(x)^2 \quad \text{for } x \ge 0 \text{ and } r \in \mathbb{R}. \tag{5.16}$$

Recall (5.1) and (5.2) that we define $\widehat{\phi}$ for $k \ge 0$. We also have

$$\begin{aligned} \widehat{\phi}(r) = &\frac{\pi^2 e^{\frac{1+k}{2}\pi i}}{\operatorname{sh}(\pi r)(\operatorname{ch}(2\pi r)+\cos(\pi k))\Gamma(\frac12-\frac k2+ir)\Gamma(\frac12-\frac k2-ir)} \\ &\cdot \left\{\cos\tfrac{k\pi}{2}\operatorname{ch}(\pi r)\left(\widetilde{\phi}(1+2ir)-\widetilde{\phi}(1-2ir)\right) - i\sin\tfrac{k\pi}{2}\operatorname{sh}(\pi r)\left(\widetilde{\phi}(1+2ir)+\widetilde{\phi}(1-2ir)\right)\right\}. \end{aligned} \tag{5.17}$$

Like [44, after (5.3)], we define $\xi_k$ as

$$\xi_k(r) := \frac{2i\pi^2 e^{\frac{1+k}{2}\pi i}}{\Gamma(\frac12-\frac k2+ir)\Gamma(\frac12-\frac k2-ir)}. \tag{5.18}$$

Then

$$\xi_k(r)\begin{cases} \ll 1 & \text{for } r\in[-1,1], \\ \asymp |r|^k e^{\pi|r|} & \text{for } r\in(-\infty,-1]\cup[1,\infty). \end{cases} \tag{5.19}$$

We refer to [52] for estimates on $J$-Bessel functions. Denote

$$F_\mu(z) := \frac{J_\mu(z)+J_{-\mu}(z)}{2\cos(\mu\pi/2)}, \quad G_\mu(z) := \frac{J_\mu(z)-J_{-\mu}(z)}{2\sin(\mu\pi/2)}.$$

As a result of the relationship $\overline{J_{2ir}(u)} = J_{-2ir}(u)$ for $r, u \in \mathbb{R}$ by [31, (10.11.9)], we have

$$F_{2ir}(u) = \frac{\operatorname{Re} J_{2ir}(u)}{\operatorname{ch}(\pi r)} \in \mathbb{R}, \quad G_{2ir}(u) = \frac{\operatorname{Im} J_{2ir}(u)}{\operatorname{sh}(\pi r)} \in \mathbb{R}.$$

Moreover, for $k \in \mathbb{Z}+\frac12$ and $k \ge 0$,

$$\widehat{\phi}(r) = \frac{\xi_k(r)\operatorname{ch}(\pi r)}{\operatorname{ch}(2\pi r)} \int_0^\infty \left(G_{2ir}(u)\cos\frac{k\pi}{2} - F_{2ir}(u)\sin\frac{k\pi}{2}\right)\frac{\phi(u)}{u}du \tag{5.20}$$

and $\widehat{\phi}(r) = \widehat{\phi}(-r)$ for $r \in \mathbb{R}$ because $F_\mu(z) = F_{-\mu}(z)$ and $G_\mu(z) = G_{-\mu}(z)$.

**Lemma 5.6.** *For $r \in [-1,1]$, uniformly and with absolute implied constants we have*

$$G_{2ir}(u) \ll \begin{cases} \ln(\frac{u}{2}), & u \in [0, \frac{3}{2}], \\ u^{-\frac{3}{2}}, & u \in [\frac{3}{2}, \infty). \end{cases} \tag{5.21}$$

*Proof.* First we deal with the range $u \in [0, \frac{3}{2}]$. The series expansion of $G_{ir}$ is given by [52, (3.9), (3.16)]:

$$\begin{aligned} G_{2ir}(u) &= \left(\frac{4r\,\mathrm{ch}(\pi r)}{\pi\,\mathrm{sh}(\pi r)}\right)^{\frac{1}{2}} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (u^2/4)^\ell \sin(2r\ln(u/2) - \phi_{2r,\ell})}{\ell! \prod_{j=0}^{\ell^2} (j+4r^2)^{1/2}} \\ &= \left(\frac{\mathrm{ch}(\pi r)}{\pi r\,\mathrm{sh}(\pi r)}\right)^{\frac{1}{2}} \sin\left(2r\ln\left(\frac{u}{2}\right) - \phi_{2r,0}\right) + O\left(\left(\frac{u}{2}\right)^2\right). \end{aligned}$$

where $\phi_{r,\ell} = \arg\Gamma(1+\ell+ir)$. The implied constant in the second equation is absolute. As a function of $r$, $\phi_{2r,0} \in C^\infty[0,1]$ and $\lim_{r\to 0}\phi_{2r,0} = 0$. Then $\phi_{2r,0}/r = O(1)$ and

$$\begin{aligned} G_{2ir}(u) &\ll r^{-1} O\left(\left|2r\ln\left(\frac{u}{2}\right)\right| + |\phi_{2r,0}|\right) + O\left(\left(\frac{u}{2}\right)^2\right) \\ &\ll \ln\left(\frac{u}{2}\right) + O(1). \end{aligned}$$

For the range $u \geq \frac{3}{2}$, we check with [52, (5.16)] where $U_s(p)$ for $s \geq 0$ are fixed polynomials of $p$ whose lowest degree term is $p^s$:

$$\begin{aligned} G_{2ir}(u) &= \left(\frac{4/\pi^2}{4r^2+u^2}\right)^{\frac{1}{4}} \left(\frac{C}{\sqrt{4r^2+u^2}} + O\left(\frac{1}{4r^2+u^2}\right)\right) \\ &\ll \left(r^2 + \frac{u^2}{4}\right)^{-\frac{3}{4}} + O\left(\left(r^2+\frac{u^2}{4}\right)^{-\frac{5}{4}}\right). \end{aligned}$$

Our claimed bound is clear as $r^2 \geq 0$. The implied constant above is absolute due to [52, (3.3)] and [53, Chapter 8, §13] or by [53, Chapter 10, (3.04)].

□

**Lemma 5.7.** *For $r \in [-1,1]$, we have*

$$|\widetilde{\phi}(1+2ir)| \ll 1, \quad |\widehat{\phi}(r)| \ll_\varepsilon (ax)^\varepsilon. \tag{5.22}$$

*Proof.* A trivial bound of $J_{2ir}$ is given by the integral representation [31, (10.9.4)]:

$$J_\nu(z) = \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_0^\pi \cos(z\cos\theta)(\sin\theta)^{2\nu} d\theta, \quad \mathrm{Re}\,\nu > -\frac{1}{2}.$$

Then we have $|J_{2ir}(u)| \leq \frac{\sqrt{\pi}}{|\Gamma(\frac{1}{2}+2ir)|}$ and

$$|\widetilde{\phi}(1+2ir)| \ll \int_{\frac{3a}{8x}}^{\frac{3a}{2x}} \frac{du}{u} \leq \ln 4 \quad \text{for } r \in [-1,1].$$

This implies that

$$\operatorname{ch}(\pi r)\int_0^\infty F_{2ir}(u)\frac{\phi(u)}{u}du \ll 1 \quad \text{for } r\in[-1,1].$$

Let the closed interval $[\alpha,\beta]=\emptyset$ when $\alpha>\beta$. With the help of (5.20), (5.19) and Lemma 5.6 we get

$$\begin{aligned}|\widehat{\phi}(r)| &\ll \frac{\operatorname{ch}(\pi r)}{\operatorname{ch}(2\pi r)}\left(\left|\int_0^\infty G_{2ir}(u)\phi(u)\frac{du}{u}\right|+\left|\int_0^\infty F_{2ir}(u)\phi(u)\frac{du}{u}\right|\right)\\ &\ll \int_{[\frac{3a}{8x},\frac{3}{2}]}\ln\left(\frac{u}{2}\right)\frac{du}{u}+\int_{[\frac{3}{2},\frac{3a}{2x}]}u^{-\frac{5}{2}}du+O(1)\\ &\ll \left(\ln\frac{3a}{16x}\right)^2+O(1)\ll (ax)^{\varepsilon}.\end{aligned}$$

The last inequality is because $a=4\pi\sqrt{\tilde{m}\tilde{n}}>0$ has a lower bound depending on $\nu$. □

When we focus on the exceptional eigenvalues $\lambda_j\in[\frac{3}{16},\frac14)$ of $\Delta_k$, recall that $\lambda_j=\frac14+r_j^2$ for $r_j\in i(0,\frac14]$. By Proposition 4.7, if we write $t_j=\operatorname{Im} r_j$, assuming $H_\theta$ (2.15) we have an upper bound $t_j\le\frac{\theta}{2}$ when $r_j\neq\frac{i}{4}$. Moreover, since the exceptional eigenvalues are discrete, we also have a largest eigenvalue less than $\frac14$, hence a lower bound $\underline{t}>0$ (depending on $N$ and $\nu$) such that $t_j\ge\underline{t}$.

**Lemma 5.8.** *With the hypothesis $H_\theta$ (2.15) for $\theta\le\frac16$, when $r=it$ and $t\in[\underline{t},\frac{\theta}{2}]$, we have*

$$\widetilde{\phi}(1\pm 2t)\ll\left(\frac{a}{x}\right)^{\pm 2t} \quad \textit{and} \quad \widehat{\phi}(r)\ll\left(\frac{a}{x}\right)^{2t}+\left(\frac{x}{a}\right)^{2t}\ll\left(\frac{a}{x}\right)^{\theta}+\left(\frac{x}{a}\right)^{\theta}. \tag{5.23}$$

*Moreover, for $r=\frac{i}{4}$ we have*

$$\widetilde{\phi}\left(1\pm\frac12\right)\ll\left(\frac{a}{x}\right)^{\pm\frac12} \quad \textit{and} \quad \widehat{\phi}\left(\frac{i}{4}\right)\ll\begin{cases}(\frac{x}{a})^{\frac12}, & k=\frac12,\\ (\frac{a}{x})^{\frac12}, & k=\frac32.\end{cases} \tag{5.24}$$

*Proof.* As in the previous lemma, when $t\in[\underline{t},\frac{\theta}{2}]$, the bound [31, (10.9.4)] gives

$$|J_{\pm 2t}(u)|\ll\frac{u^{\pm 2t}}{\Gamma(\frac12-\theta)} \quad \text{and} \quad |\widetilde{\phi}(1\pm 2t)|\ll\int_{\frac{3a}{8x}}^{\frac{3a}{2x}}u^{\pm 2t}\frac{du}{u}\ll\left(\frac{a}{x}\right)^{\pm 2t}.$$

The bound for $\widehat{\phi}$ follows from (5.17). When $r=\frac{i}{4}$, by [31, (10.16.1)] we have

$$J_{-\frac12}(u)\ll u^{-\frac12} \quad \text{and} \quad J_{\frac12}(u)\ll u^{-\frac12}\sin u\le u^{\frac12}.$$

The bounds for $\widetilde{\phi}(1\pm\frac12)$ and $\widehat{\phi}(\frac{i}{4})$ follow from the same process above with (5.1) and (5.3). □

For the range $|r|\ge 1$ we have

**Lemma 5.9.** *[44, Lemma 6.3] Let $k=\frac12$ or $\frac32$. Then*

$$\widehat{\phi}(r)\ll\begin{cases}r^{k-\frac32}, & r\ge 1,\\ r^k\min(r^{-\frac32},r^{-\frac52}\frac{x}{T}), & r\ge\max(\frac{a}{x},1).\end{cases} \tag{5.25}$$

*Remark.* In the original paper they stated the result for $k=\pm\frac12$. However, the power $r^k$ in the estimate above only arises from $\xi_k(r)e^{-\pi|r|}$ (5.18) and by (5.19) we get the above lemma for weight $k=\frac32$.

### 5.2.1 A special test function

Here we choose the special test function $\phi$ again which satisfies Setting 4.9 to compute the terms corresponding to the exceptional spectrum $r \in i(0, \frac{1}{4}]$ in Theorem 1.7. However, in this case we focus on weight $k = \frac{1}{2}$ or $\frac{3}{2}$. Let $\lambda \in [\frac{3}{16}, \frac{1}{4})$ be an exceptional eigenvalue of $\Delta_k$ on $\Gamma_0(N)$, we set $\lambda = s(1-s)$ for $s \in (\frac{1}{2}, \frac{3}{4}]$ and

$$t = \operatorname{Im} r = \sqrt{\tfrac{1}{4} - \lambda} = \sqrt{\tfrac{1}{4} - s(1-s)} = s - \tfrac{1}{2}.$$

Since the exceptional spectrum is discrete, let the lower bound for $t > 0$ be $\underline{t}$ depending on $N$ and $\nu$. Recall Setting 4.8. Let $0 < T' \leq T \leq \frac{x}{3}$ be $T' := Tx^{-\delta} \asymp x^{1-2\delta}$. We choose $\phi$ as in Setting 4.12.

Now we take $r = it \in i(0, \frac{1}{4}]$. When $u \leq 1.999$, by the series expansion [31, (10.2.2)]:

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+1+\nu)} \left(\frac{z}{2}\right)^{2j},$$

we have

$$J_{\pm 2t}(u) = \frac{(u/2)^{\pm 2t}}{\Gamma(1 \pm 2t)} + O\left(\left(\frac{u}{2}\right)^{2 \pm 2t}\right), \qquad 0 < u \leq 1.999. \tag{5.26}$$

The implied constant is absolute. Now we compute the bound for $\widetilde{\phi}$ and $\widehat{\phi}$.

**Lemma 5.10.** *Assuming $H_\theta$ (2.15) for $\theta \leq \frac{1}{6}$ and with the choice of $\phi$ in Setting 4.12, when $r = it \in i(0, \frac{1}{4}]$,*

$$\begin{aligned} \widetilde{\phi}(1-2t) &= \frac{1}{\Gamma(1-2t)} \int_{\frac{a}{2x}}^{\frac{a}{x}} \left(\frac{u}{2}\right)^{-2t} \frac{\phi(u)}{u} du + O\left(a^{-2t} x^{2t-\delta} + 1\right) \\ &= \frac{2^{2t}(2^{2t}-1)}{2t\Gamma(1-2t)} \left(\frac{x}{a}\right)^{2t} + O\left(a^{-2t} x^{2t-\delta} + 1\right), \end{aligned} \tag{5.27}$$

*Proof.* When $1.999 < \frac{a}{x-T} \leq \frac{3a}{2x}$, we get $x \ll a$ and $\widetilde{\phi}(1-2t) = O(1)$ by Lemma 5.8, so the lemma is true in this case. When $\frac{a}{x-T} \leq 1.999$, we have $a \ll x$ and with (5.26),

$$\begin{aligned} \widetilde{\phi}(1-2t) &= \int_0^\infty \frac{(u/2)^{-2t}}{\Gamma(1-2t)} \frac{\phi(u)}{u} du + O\left(\int_0^\infty \left(\frac{u}{2}\right)^{2-2t} \frac{\phi(u)}{u} du\right) \\ &= \frac{2^{2t}}{\Gamma(1-2t)} \int_{\frac{a}{2x}}^{\frac{a}{x}} u^{-2t-1} du + \frac{2^{2t}}{\Gamma(1-2t)} \left(\int_{\frac{a}{2x+2T}}^{\frac{a}{2x}} + \int_{\frac{a}{x}}^{\frac{a}{x-T}}\right) u^{-2t-1}\phi(u) du \\ &\quad + O\left(\int_0^\infty u^{1-2t}\phi(u) du\right) \\ &=: \frac{2^{2t}(2^{2t}-1)}{2t\Gamma(1-2t)} \left(\frac{x}{a}\right)^{2t} + (I_1 + I_2) + O(I_3). \end{aligned}$$

Recall that we always have the lower bound $\underline{t} > 0$ for $t = \operatorname{Im} r$. A bound for $I_1$ and $I_2$ follows from the same process as [14, Proof of Lemma 7.2]:

$$I_1 + I_2 \ll \left(\int_{\frac{a}{2x+2T}}^{\frac{a}{2x}} + \int_{\frac{a}{x}}^{\frac{a}{x-T}}\right) u^{-2t-1}\phi(u) du \ll a^{-2t} x^{2t-\delta}.$$

We also get

$$I_3 \ll \int_{\frac{3a}{8x}}^{\frac{3a}{2x}} u^{1-2t} du \ll \left(\frac{a}{x}\right)^{2-2t} \ll 1$$

and finish the proof. $\square$

**Lemma 5.11.** *Assume $H_\theta$ (2.15) for $\theta \le \frac{1}{6}$. For $r = it \in i(0, \frac{\theta}{2}]$ we have*

$$\widehat{\phi}(r) = \frac{e^{\frac{k\pi i}{2}}\cos(\pi t)\Gamma(\frac{1}{2}+\frac{k}{2}+t)\Gamma(2t)}{\Gamma(\frac{1}{2}-\frac{k}{2}+t)2^{2t}\pi^{2t}(\tilde{m}\tilde{n})^t} \cdot \frac{(2^{2t}-1)x^{2t}}{2t} + O\left(\frac{x^{2t-\delta}}{a^{2t}} + \frac{a^{2t}}{x^{2t}} + 1\right).$$

*Moreover,*

$$\widehat{\phi}(\tfrac{i}{4}) = \begin{cases} 2e^{\frac{\pi i}{4}}(\sqrt{2}-1)(\frac{x}{a})^{\frac{1}{2}} + O(x^{-\delta}(\frac{x}{a})^{\frac{1}{2}}+1) & \text{for } k = \frac{1}{2}, \\ e^{\frac{3\pi i}{4}}(1-\frac{1}{\sqrt{2}})(\frac{a}{x})^{\frac{1}{2}} + O(x^{-\delta}(\frac{a}{x})^{\frac{1}{2}}+1) & \text{for } k = \frac{3}{2}. \end{cases}$$

*The implied constants only depend on $N$ and $\nu$.*

*Proof.* When $t \in [\underline{t}, \frac{\theta}{2}]$, we substitute Lemma 5.10 into (5.2) and use Lemma 5.8 to get

$$\begin{aligned} \widehat{\phi}(it) &= \frac{i\pi^2 e^{\frac{k\pi i}{2}}\left(\cos(\frac{k\pi}{2}-\pi t)\widetilde{\phi}(1-2t) - \cos(\frac{k\pi}{2}+\pi t)\widetilde{\phi}(1+2t)\right)}{i\sin(\pi t)\cos(2\pi t)\Gamma(\frac{1}{2}-\frac{k}{2}-t)\Gamma(\frac{1}{2}-\frac{k}{2}+t)} \\ &= \frac{\pi^2 e^{\frac{k\pi i}{2}}\cos(\frac{k\pi}{2}-\pi t)2^{2t}(2^{2t}-1)(x/a)^{2t}}{\sin(\pi t)\cos(2\pi t)\Gamma(\frac{1}{2}-\frac{k}{2}-t)\Gamma(\frac{1}{2}-\frac{k}{2}+t)2t\Gamma(1-2t)} + O\left(\frac{x^{2t-\delta}}{a^{2t}} + \frac{a^{2t}}{x^{2t}} + 1\right). \end{aligned}$$

With the help of the functional equation of the $\Gamma$ function

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad \text{for } z \in \mathbb{C}\setminus\mathbb{Z}$$

and the trigonometric identities

$$\sin(\tfrac{\pi}{2}-x) = \cos x, \quad 2\cos x\cos y = \cos(x+y)+\cos(x-y) \quad \text{for } x, y \in \mathbb{R},$$

we have

$$\begin{aligned} \frac{\pi}{\sin(\pi t)\Gamma(1-2t)} &= 2\cos(\pi t)\Gamma(2t), \\ \frac{\pi}{\Gamma(\frac{1}{2}-\frac{k}{2}-t)} &= \Gamma(\tfrac{1}{2}+\tfrac{k}{2}+t)\cos(\tfrac{k\pi}{2}+\pi t), \\ \text{and} \quad 2\cos(\tfrac{k\pi}{2}-\pi t)\cos(\tfrac{k\pi}{2}+\pi t) &= \cos(2\pi t). \end{aligned}$$

Then the first part of the lemma follows. The implied constant only depends on $N$ and $\nu$ because $t \in [\underline{t}, \frac{\theta}{2}]$ is bounded above and below away from 0.

When $t = \frac{1}{4}$, the process is similar to the proof of Lemma 5.10 with the help of (5.3). First we deal with the case $k = \frac{1}{2}$ with $\cos u = 1 + O(u^2)$ for $u \in [0, \frac{\pi}{2}]$. Thus, when $\frac{a}{x-T} > \frac{\pi}{2}$, we have $x \ll a$ and $\widehat{\phi}(\frac{i}{4}) = O(1)$

in this case. When $\frac{a}{x-T} \le \frac{\pi}{2}$, we have $a \ll x$ and

$$\begin{aligned}\widehat{\phi}(\tfrac{i}{4}) &= e^{\frac{\pi i}{4}} \int_0^\infty \phi(u) u^{-\frac{3}{2}} du + O\left(\int_0^\infty \phi(u) u^{\frac{1}{2}} du\right) \\ &= e^{\frac{\pi i}{4}} \int_{\frac{a}{2x}}^{\frac{a}{x}} u^{-\frac{3}{2}} du + e^{\frac{\pi i}{4}} \left(\int_{\frac{a}{2x+2T}}^{\frac{a}{2x}} + \int_{\frac{a}{x}}^{\frac{a}{x-T}}\right) u^{-\frac{3}{2}} \phi(u) du + O(1) \\ &= e^{\frac{\pi i}{4}} (2\sqrt{2}-2) \left(\frac{x}{a}\right)^{\frac{1}{2}} + O\left(x^{-\delta}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right) + O(1).\end{aligned}$$

The case for $k = \frac{3}{2}$ is similar using $\sin u = u + O(u^3)$ for $u \in [0, \frac{\pi}{2}]$. □

## 5.3 Proof of Theorem 1.7, same-sign case

The proof depends on the following two propositions for the Fourier coefficients of Maass forms, which were originally obtained for the discrete spectrum in [47, Theorem 4.1] and [12, Theorem 4.3]. The author proved the generalized propositions in §4.4 to include the continuous spectrum. Recall our notations in Settings 4.8 and 4.9. Recall Proposition 4.14 and Proposition 4.15. Before we start the proof, we need to make a few remarks about the weight $k$:

(1) The trace formula (Theorem 5.1) works for $k = \frac{1}{2}$ and $\frac{3}{2}$.

(2) The estimates on $\widehat{\phi}$ and $\widetilde{\phi}$ in the previous section work for $k = \frac{1}{2}$ and $\frac{3}{2}$.

(3) Proposition 4.14 and Proposition 4.15 work for $k = \frac{1}{2}$ and $-\frac{1}{2}$.

Therefore, in this section, we separate the proof of Theorem 1.7 into two cases $k = \frac{1}{2}$ and $-\frac{1}{2}$. In the second case we will apply the Maass lowering operator $L_{\frac{3}{2}}$ (2.11) to connect the eigenforms of weight $\frac{3}{2}$ and weight $-\frac{1}{2}$.

We declare that all implicit constants for the bounds in this section depend on $N$, $\nu$ and $\varepsilon$, and we drop the subscripts unless specified.

Since the exceptional spectral parameter $r_j \in i(0, \frac{1}{4}]$ of the Laplacian $\Delta_k$ on $\Gamma = \Gamma_0(N)$ is discrete, $t_j = \operatorname{Im} r_j$ has a positive lower bound denoted as $\underline{t} > 0$ depending on $N$ and $\nu$. We also have $2 \operatorname{Im} r_\Delta \le \theta$ assuming $H_\theta$ (2.15) by Theorem 4.7. Recall the definition of $A(m,n)$ and $A_u(m,n)$ in (4.43).

$$\begin{aligned}A(m,n) &:= (\tilde{m}^{\frac{131}{294}} + u_m)^{\frac{1}{2}} (\tilde{n}^{\frac{131}{294}} + u_n)^{\frac{1}{2}} \ll (\tilde{m}\tilde{n})^{\frac{131}{588}} + \tilde{m}^{\frac{131}{588}} u_n^{\frac{1}{2}} + \tilde{n}^{\frac{131}{588}} u_m^{\frac{1}{2}} + (u_m u_n)^{\frac{1}{2}}, \\ A_u(m,n) &:= A(m,n)^{\frac{1}{4}} (\tilde{m}\tilde{n})^{\frac{3}{16}} \ll (\tilde{m}\tilde{n})^{\frac{143}{588}} + \tilde{m}^{\frac{143}{588}} \tilde{n}^{\frac{3}{16}} u_n^{\frac{1}{8}} + \tilde{m}^{\frac{3}{16}} u_m^{\frac{1}{8}} \tilde{n}^{\frac{143}{588}} + (\tilde{m}\tilde{n})^{\frac{3}{16}} (u_m u_n)^{\frac{1}{8}}.\end{aligned}$$

Recall the notations in Setting 4.8 and Setting 4.9. The following inequalities will be used later in the proof:

$$A(m,n) \ll A_u(m,n) \ll (\tilde{m}\tilde{n})^{\frac{1}{4}}; \tag{5.28}$$

$$\left(\frac{a}{x}\right)^{\beta} A(m,n) \ll A_u(m,n) \quad \text{for } 0 \le \beta \le \frac{3}{2}, \quad \text{when } x \gg A_u(m,n)^2. \tag{5.29}$$

### 5.3.1 On the case $k = \frac{1}{2}$

Let $\rho_j(n)$ denote the coefficients of an orthonormal basis $\{v_j(\cdot)\}$ of $\tilde{\mathcal{L}}_{\frac{1}{2}}(N, \nu)$. For each singular cusp $\mathfrak{a}$ of $\Gamma = \Gamma_0(N)$, let $\rho_{\mathfrak{a}}(n, r)$ be defined as in (4.3). Recall the definition of $\tau_j(m,n)$ in Theorem 1.7 and the notations in Settings 4.8 and 4.9. We claim the following proposition:

**Proposition 5.12.** *With the same setting as Theorem 1.7 for $k=\frac{1}{2}$, when $2x \geq A_u(m,n)^2$, we have*

$$\sum_{\substack{x<c\leq 2x\\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} \ll \left(x^{\frac16}+A_u(m,n)\right)(\tilde{m}\tilde{n}x)^{\varepsilon}. \tag{5.30}$$

We first show that Proposition 5.12 implies Theorem 1.7 in the case $k=\frac12$, which follows from a similar process as [14, after Proposition 9.1]. Recall that $2\operatorname{Im} r_j = 2s_j-1$ for $r_j\in i(0,\frac14]$ and that the corresponding exceptional eigenvalue $\lambda_j=\frac14+r_j^2=s_j(1-s_j)$. The sum to be estimated is

$$\sum_{N|c\leq X} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]} \tau_j(m,n)\frac{X^{2\operatorname{Im} r_j}}{2\operatorname{Im} r_j}, \tag{5.31}$$

where

$$\tau_j(m,n) = 2i^{\frac12}\overline{\rho_j(m)}\rho_j(n)\pi^{1-2s_j}(4\tilde{m}\tilde{n})^{1-s_j}\frac{\Gamma(s_j+\frac14)\Gamma(2s_j-1)}{\Gamma(s_j-\frac14)}.$$

Since $t_j=\operatorname{Im} r_j\in[\underline{t},\frac14]$ and $s_j=\operatorname{Im} r_j+\frac12\in[\underline{t}+\frac12,\frac34]$, the quantity

$$\pi^{1-2s_j}4^{1-s_j}\frac{\Gamma(s_j+\frac14)\Gamma(2s_j-1)}{\Gamma(s_j-\frac14)}$$

is bounded from above and below. By Proposition 4.15,

$$\frac{\tau_j(m,n)}{2s_j-1} \ll |\rho_j(m)\rho_j(n)|(\tilde{m}\tilde{n})^{1-s_j} \ll A(m,n)(\tilde{m}\tilde{n})^{\frac12-s_j+\varepsilon}. \tag{5.32}$$

When $X\ll A_u(m,n)^2$, since $A(m,n)\ll(\tilde{m}\tilde{n})^{\frac14}$ by (5.28),

$$\begin{aligned}\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1} &\ll A(m,n)|\tilde{m}\tilde{n}|^{\frac12-s_j+\varepsilon}A_u(m,n)^{4s_j-2}\\ &= A(m,n)^{s_j+\frac12}|\tilde{m}\tilde{n}|^{\frac18-\frac14 s_j+\varepsilon} \ll A_u(m,n)(\tilde{m}\tilde{n})^{\varepsilon}.\end{aligned} \tag{5.33}$$

So in this case we get Theorem 1.7 where the $\tau_j$ terms are absorbed in the errors.

When $X\geq A_u(m,n)^2$, the segment for summing Kloosterman sums on $1\leq c\leq A_u(m,n)^2$ contributes a $O_{\nu,\varepsilon}(A_u(m,n)|\tilde{m}\tilde{n}|^{\varepsilon})$ by condition (2) of Definition 1.6. The segment for $A_u(m,n)^2\leq c\leq X$ can be broken into no more than $O(\log X)$ dyadic intervals $x<c\leq 2x$ with $A_u(m,n)^2\leq x\leq\frac{X}{2}$ and we use Proposition 5.12 for both the Kloosterman sum and the $\tau_j$ terms. In summing dyadic intervals, for each $r_j\in i(0,\frac14]$, we get

$$\begin{aligned}&\sum_{\ell=1}^{\lceil\log_2(X/A_u(m,n)^2)\rceil}\frac{(2^{2s_j-1}-1)\tau_j(m,n)}{2s_j-1}\left(\frac{X}{2^\ell}\right)^{2s_j-1}\\ &\qquad= \frac{\tau_j(m,n)}{2s_j-1}X^{2s_j-1}\left(1-2^{(1-2s_j)\lceil\log_2(X/A_u(m,n)^2)\rceil}\right).\end{aligned}$$

The difference between the above quantity and the quantity $\tau_j(m,n)\dfrac{X^{2s_j-1}}{2s_j-1}$ in (5.31) is

$$\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}\cdot 2^{(1-2s_j)\lceil \log_2\left(X/A_u(m,n)^2\right)\rceil} \ll \frac{\tau_j(m,n)}{2s_j-1}A_u(m,n)^{4s_j-2} \ll A_u(m,n) \tag{5.34}$$

by (5.32). In conclusion, for $X \geq A_u(m,n)^2$ we get

$$\begin{aligned}
&\sum_{N|c\leq X}\frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]}\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1}\\
&= \sum_{A_u(m,n)^2<c\leq X}\frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]}\tau_j(m,n)\frac{X^{2s_j-1}}{2s_j-1} + O(A_u(m,n)|\tilde m\tilde n|^{\varepsilon})\\
&= \sum_{\ell=1}^{\lceil \log_2\left(X/A_u(m,n)^2\right)\rceil}\left(\sum_{\frac{X}{2^\ell}<c\leq\frac{X}{2^{\ell-1}}}\frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac14]}\frac{(2^{2s_j-1}-1)\tau_j(m,n)}{2s_j-1}\left(\frac{X}{2^\ell}\right)^{2s_j-1}\right)\\
&\quad + O(A_u(m,n)|\tilde m\tilde n|^{\varepsilon})\\
&\ll \left(X^{\frac16}+A_u(m,n)\right)|\tilde m\tilde n X|^{\varepsilon}
\end{aligned}$$

where the second equality follows from (5.34) and the last inequality is by Proposition 5.12. Theorem 1.7 follows in the case $k=\frac12$.

The proof of Proposition 5.12 takes the rest of this subsection. For $r_j\in i(0,\frac14]$, by Proposition 4.15 we have

$$\sqrt{\tilde m\tilde n}\,\overline{\rho_j(m)}\rho_j(n) \ll A(m,n)(\tilde m\tilde n)^{\varepsilon}.$$

Recall that $a=4\pi\sqrt{\tilde m\tilde n}$ and $\delta=\frac13$ in Setting 4.8. Thanks to $H_{\frac{7}{64}}$ (2.15) and Proposition 4.7, when $r_j=it_j\in i(0,\frac{\theta}{2}]$ we have $2t_j<\delta=\frac13$. Since $2x\geq A_u(m,n)^2$ by hypothesis, it follows from (5.29) that

$$\sqrt{\tilde m\tilde n}\,\overline{\rho_j(m)}\rho_j(n)\left(\frac{x^{2t_j-\delta}}{a^{2t_j}}+\frac{a^{2t_j}}{x^{2t_j}}+1\right) \ll A_u(m,n)(\tilde m\tilde n)^{\varepsilon}.$$

Applying Lemma 5.11 where $t_j\in[\underline{t},\frac{\theta}{2}]$ and recalling the definition of $\tau_j$ in Theorem 1.7, we get

$$\begin{aligned}
&4\sqrt{\tilde m\tilde n}\,\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\hat\phi(r_j)\\
&\qquad = (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} + O\left(A_u(m,n)(\tilde m\tilde n)^{\varepsilon}\right).
\end{aligned}\tag{5.35}$$

When $r_j=\frac{i}{4}$ and $k=\frac12$, Lemma 5.11 and (5.29) give

$$4\sqrt{\tilde m\tilde n}\,\frac{\overline{\rho_j(m)}\rho_j(n)}{\cos\frac{\pi}{4}}\hat\phi(\tfrac{i}{4}) = 2(\sqrt2-1)\tau_j(m,n)x^{\frac12} + O\left(x^{\frac12-\delta}(\tilde m\tilde n)^{\varepsilon}\right). \tag{5.36}$$

With the help of (5.35) and (5.36) we break up the left hand side of (5.30) to obtain the following analogue to [14, (9.8)]:

$$
\begin{aligned}
&\left|\sum_{\substack{x<c\leq 2x\\ N|c}}\frac{S(m,n,c,\nu)}{c}-\sum_{r_j\in i(0,\frac14]}(2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}\right|\\
&\leq\left|\sum_{\substack{x<c\leq 2x\\ N|c}}\frac{S(m,n,c,\nu)}{c}-\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right)\right|+O\left(\left(x^{\frac12-\delta}+A_u(m,n)\right)(\tilde m\tilde n)^{\varepsilon}\right)\\
&\quad+\left|\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right)-4\sqrt{\tilde m\tilde n}\sum_{r_j\in i(0,\frac14]}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\\
&=:S_1+O\left(\left(x^{\frac12-\delta}+A_u(m,n)\right)(\tilde m\tilde n)^{\varepsilon}\right)+S_2.
\end{aligned}\tag{5.37}
$$

The first sum $S_1$ above can be estimated by condition (2) of Definition 1.6 as

$$
S_1\leq\sum_{\substack{x-T\leq c\leq x\\ 2x\leq c\leq 2x+2T\\ N|c}}\frac{|S(m,n,c,\nu)|}{c}\ll_{N,\nu,\delta,\varepsilon}x^{\frac12-\delta}(\tilde m\tilde n x)^{\varepsilon}.\tag{5.38}
$$

We then prove a bound for $S_2$. By Theorem 5.1, we have

$$
S_2\ll|\mathcal{U}_{\frac12}|+\left|\sqrt{\tilde m\tilde n}\sum_{r_j\geq 0}\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)+\sqrt{\tilde m\tilde n}\sum_{\text{singular }\mathfrak{a}}\int_{-\infty}^{\infty}\overline{\rho_{\mathfrak a}(m,r)}\rho_{\mathfrak a}(n,r)\frac{\widehat{\phi}(r)}{\operatorname{ch}\pi r}dr\right|.\tag{5.39}
$$

#### 5.3.1.1 Contribution from holomorphic forms

For $k=\frac12$ or $\frac32$, recall the notation $\mathscr{B}_k$ before Theorem 5.1. For $l\geq 1$, let $\{F_{j,l}(\cdot)\}_j$ be an orthonormal basis of $S_{k+2l}(N,\nu)$ with Fourier coefficient $a_{F,j,l}$. By Proposition 5.5, uniformly for every $l\geq 1$ with $d_l:=\dim S_{k+2l}(N,\nu)$, we have $k+2l\geq\frac52$ and

$$
\begin{aligned}
&\frac{\Gamma(k+2l-1)}{(4\pi)^{k+2l-1}(\tilde m\tilde n)^{\frac{k+2l-1}{2}}}\sum_{j=1}^{d_l}\overline{a_{F,j,l}(m)}a_{F,j,l}(n)\\
&\leq\left(\frac{\Gamma(k+2l-1)}{(4\pi\tilde n)^{k+2l-1}}\sum_{j=1}^{d_l}|a_{F,j,l}(m)|^2\right)^{\frac12}\left(\frac{\Gamma(k+2l-1)}{(4\pi\tilde m)^{k+2l-1}}\sum_{j=1}^{d_l}|a_{F,j,l}(n)|^2\right)^{\frac12}\\
&\ll(\tilde m^{\frac{19}{42}}+u_m)^{\frac12}(\tilde n^{\frac{19}{42}}+u_n)^{\frac12}(\tilde m\tilde n)^{\varepsilon}.
\end{aligned}
$$

We also have

$$
\sum_{l=1}^{\infty}(k+2l-1)\,|\widetilde{\phi}(k+2l)|\ll 1+\frac{a}{x}
$$

by [20, Lemma 5.1 and proof of Lemma 7.1] and Lemma 5.8. Note that [20, Lemma 5.1] is only for $k = \frac{1}{2}$, while the same process works for $k = \frac{3}{2}$. Then the contribution from $\mathcal{U}_k$ is

$$\begin{aligned}\mathcal{U}_k &= \sum_{l=1}^{\infty} \frac{k+2l-1}{4\pi} \tilde{\phi}(k+2l) \frac{\Gamma(k+2l-1)}{(4\pi)^{k+2l-1}(\tilde{m}\tilde{n})^{\frac{k+2l-1}{2}}} \sum_{j=1}^{d_l} \overline{a_{F,j,l}(m)} a_{F,j,l}(n) \\ &\ll \left(1 + \frac{a}{x}\right) (\tilde{m}^{\frac{19}{42}} + u_m)^{\frac{1}{2}} (\tilde{n}^{\frac{19}{42}} + u_n)^{\frac{1}{2}} (\tilde{m}\tilde{n})^{\varepsilon}.\end{aligned}$$

Recall $a = 4\pi\sqrt{\tilde{m}\tilde{n}}$ and (4.43) for the definition of $A_u(m,n)$. Since

$$\tilde{n}^{\frac{19}{42}} \ll \tilde{n}^{\frac{131}{294}\cdot\frac{1}{4}+\frac{3}{8}} \leq (\tilde{n}^{\frac{131}{294}} + u_n)^{\frac{1}{4}} \tilde{n}^{\frac{3}{8}} \quad \text{and} \quad u_n \ll u_n^{\frac{1}{4}} \tilde{n}^{\frac{3}{8}} \leq (\tilde{n}^{\frac{131}{294}} + u_n)^{\frac{1}{4}} \tilde{n}^{\frac{3}{8}},$$

we get $(\tilde{m}^{\frac{19}{42}} + u_m)^{\frac{1}{2}} (\tilde{n}^{\frac{19}{42}} + u_n)^{\frac{1}{2}} \ll A_u(m,n)$. Moreover, since

$$\tilde{n}^{\frac{19}{42}+1} \ll \tilde{n}^{\frac{131}{294}\cdot\frac{3}{4}+\frac{9}{8}} \leq (\tilde{n}^{\frac{131}{294}} + u_n)^{\frac{3}{4}} \tilde{n}^{\frac{9}{8}}, \qquad n u_n \ll u_n^{\frac{3}{4}} \tilde{n}^{\frac{9}{8}} \leq (\tilde{n}^{\frac{131}{294}} + u_n)^{\frac{3}{4}} \tilde{n}^{\frac{9}{8}},$$

and $2x \geq A_u(m,n)^2$ by hypothesis, we also get

$$(\tilde{m}^{\frac{19}{42}} + u_m)^{\frac{1}{2}} (\tilde{n}^{\frac{19}{42}} + u_n)^{\frac{1}{2}} \cdot \frac{a}{x} \ll (\tilde{m}^{\frac{19}{42}} + u_m)^{\frac{1}{2}} (\tilde{n}^{\frac{19}{42}} + u_n)^{\frac{1}{2}} \cdot \frac{a}{A_u(m,n)^2} \ll A_u(m,n).$$

Finally we conclude

$$\mathcal{U}_k \ll A_u(m,n)(\tilde{m}\tilde{n})^{\varepsilon} \qquad \text{for } k = \tfrac{1}{2} \text{ or } \tfrac{3}{2}. \tag{5.40}$$

#### 5.3.1.2 Contribution from Maass cusp forms and Eisenstein series.

We combine the two propositions at the beginning of this section and bounds on $\widehat{\phi}$ in Section 4 to estimate the contribution from the remaining part of $S_2$ (5.39) other than $\mathcal{U}_k$. The process is the same as §4.5 for $|r| \leq 1$ as $\widehat{\phi}$ shares the same bound as $\check{\phi}$ there. We record the bounds in the following equations.

Fix $k = \frac{1}{2}$. In the following estimations we focus on the discrete spectrum $r_j \geq 0$ because each bound for $r_j \in [a,b]$ for any interval $[a,b] \subset \mathbb{R}$ is the same as the bound for $r \in [a,b] \cup [-b,-a]$ in the continuous spectrum. This is a direct result from Proposition 4.14 and Proposition 4.15. Recall that $2x \geq A_u(m,n)^2$ in the assumption of Proposition 5.12.

For $r \in [0,1)$, we apply Lemma 5.7, Proposition 4.15 and Cauchy-Schwarz to get

$$\sqrt{\tilde{m}\tilde{n}} \sum_{r \in [0,1)} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \widehat{\phi}(r_j) \right| \ll A(m,n)(\tilde{m}\tilde{n}x)^{\varepsilon}. \tag{5.41}$$

For $r \in [1, \frac{a}{x})$, we apply Proposition 4.15 and $\widehat{\phi}(r) \ll r^{-1}$ from (5.25). Since

$$S(R) := \sqrt{\tilde{m}\tilde{n}} \sum_{r \in [1,R]} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch} \pi r_j} \right| \ll A(m,n) R^{\frac{5}{2}} (\tilde{m}\tilde{n})^{\varepsilon}, \tag{5.42}$$

with the help of (5.29) we have

$$\begin{aligned}\sqrt{\tilde m\tilde n}\sum_{r\in[1,\frac{a}{x})}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right| &\ll r^{-1}S(r)\Big|_{r=1}^{\frac{a}{x}}+\int_1^{\frac{a}{x}}S(r)r^{-2}dr\\ &\ll A(m,n)\left(\frac{a}{x}\right)^{\frac32}(\tilde m\tilde n x)^{\varepsilon}\ll A_u(m,n)(\tilde m\tilde n x)^{\varepsilon}.\end{aligned}\tag{5.43}$$

Let

$$P(m,n):=2(\tilde m\tilde n)^{\frac18}A(m,n)^{-\frac12}\ge 1.$$

Divide $r\ge\max(\frac{a}{x},1)$ into two parts: $\max\left(\frac{a}{x},1\right)\le r<P(m,n)$ and $r\ge\max\left(\frac{a}{x},1,P(m,n)\right)$. We apply Proposition 4.15 on the first range and $\widehat{\phi}(r)\ll r^{-1}$ from (5.25) to get

$$\sqrt{\tilde m\tilde n}\sum_{\max(\frac{a}{x},1)\le r_j<P(m,n)}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\ll A_u(m,n)(\tilde m\tilde n x)^{\varepsilon}\tag{5.44}$$

by partial summation as in (5.43). We divide the second range into dyadic intervals $C\le r_j<2C$. Applying Proposition 4.14 with $\beta=\frac12+\varepsilon$ and $\widehat{\phi}(r)\ll\min(r^{-1},r^{-2}\frac{x}{T})$ from (5.25), we get

$$\begin{aligned}&\sqrt{\tilde m\tilde n}\sum_{C\le r_j<2C}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\\ &\ll\min\left(C^{-1},C^{-2}\frac{x}{T}\right)C^{-\frac12}\left(C^2+(\tilde m^{\frac14}+\tilde n^{\frac14})C+(\tilde m\tilde n)^{\frac14}\right)(\tilde m\tilde n x)^{\varepsilon}\\ &\ll\left(\min\left(C^{\frac12},C^{-\frac12}\frac{x}{T}\right)+(\tilde m^{\frac14}+\tilde n^{\frac14})C^{-\frac12}+(\tilde m\tilde n)^{\frac14}C^{-\frac32}\right)(\tilde m\tilde n x)^{\varepsilon}.\end{aligned}\tag{5.45}$$

Next we sum over dyadic intervals. For the first term $\min(C^{\frac12},C^{-\frac12}\frac{x}{T})$, when

$$\min\left(C^{\frac12},C^{-\frac12}\frac{x}{T}\right)=C^{\frac12}:\qquad\sum_{\substack{j\ge1:\ 2^jC=\frac{x}{T}\\ C\ge P(m,n)}}C^{\frac12}\le\sum_{j=1}^{\infty}2^{-\frac{j}{2}}\left(\frac{x}{T}\right)^{\frac12}\ll\left(\frac{x}{T}\right)^{\frac12},$$

and when

$$\min\left(C^{\frac12},C^{-\frac12}\frac{x}{T}\right)=C^{-\frac12}\frac{x}{T}:\qquad\sum_{j\ge0:\ C=2^j\frac{x}{T}}C^{-\frac12}\frac{x}{T}\le\sum_{j=0}^{\infty}2^{-\frac{j}{2}}\left(\frac{x}{T}\right)^{\frac12}\ll\left(\frac{x}{T}\right)^{\frac12}.$$

So after summing up from (5.45), recalling $T\asymp x^{1-\delta}$ in Setting 4.8, using $C\ge P(m,n)$ and (5.28), we have

$$\begin{aligned}&\sqrt{\tilde m\tilde n}\sum_{r_j\ge\max(\frac{a}{x},1,P(m,n))}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\\ &\ll\left(\left(\frac{x}{T}\right)^{\frac12}+(\tilde m+\tilde n)^{\frac14}(\tilde m\tilde n)^{-\frac1{16}}A(m,n)^{\frac14}+(\tilde m\tilde n)^{\frac1{16}}A(m,n)^{\frac34}\right)(\tilde m\tilde n x)^{\varepsilon}\\ &\ll\left(x^{\frac{\delta}{2}}+(\tilde m\tilde n)^{\frac{3}{16}}A(m,n)^{\frac14}\right)(\tilde m\tilde n x)^{\varepsilon}.\end{aligned}\tag{5.46}$$

Combining (5.44) and (5.46) we have

$$\sqrt{\tilde m\tilde n}\sum_{r_j\ge\max(\frac{a}{x},1)}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\ll\left(x^{\frac{\delta}{2}}+A_u(m,n)\right)(\tilde m\tilde n x)^{\varepsilon}. \tag{5.47}$$

From (5.37), (5.38), (5.39), (5.40), (5.41), (5.43), and (5.47), we get

$$\sum_{\substack{x<c\le 2x\\ N|c}}\frac{S(m,n,c,\nu)}{c}-\sum_{r_j\in i(0,\frac14]}(2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}$$
$$\ll\left(x^{\frac12-\delta}+x^{\frac{\delta}{2}}+A_u(m,n)\right)(\tilde m\tilde n x)^{\varepsilon}.$$

Proposition 5.12 follows by choosing $\delta=\frac13$. We finish the proof of Theorem 1.7 in weight $\frac12$.

### 5.3.2 On the case $k=-\frac12$

Recall the remark after Proposition 4.15. Let $\rho_j'(n)$ denote the Fourier coefficients of an orthonormal basis $\{v_j'(\cdot)\}$ of $\tilde{\mathcal{L}}_{\frac32}(N,\nu)$. For each singular cusp $\mathfrak{a}$ of $(\Gamma,\nu)$, let $E_{\mathfrak a}'(\cdot,s)$ be the associated Eisenstein series in weight $\frac32$. Let $\rho_{\mathfrak a}'(n,r)$ be defined as in (4.3) associated with $E_{\mathfrak a}'(z,\frac12+ir)$ for $r\in\mathbb{R}$.

Recall the definition of the Maass lowering operator $L_k$ in (2.11) and $H_\theta$ (2.15) for $\theta=\frac{7}{64}$. By [54, (4.52)] (where they used $\Lambda_k$ for the lowering operator and $\lambda(s)=s(1-s)$), the set

$$\left\{v_j:=\left(\tfrac{1}{16}+r_j^2\right)^{-\frac12}L_{\frac32}v_j':\ r_j\ne\tfrac{i}{4}\right\}\text{ is an orthonormal basis of }\bigoplus_{r_j\ne\frac{i}{4}}\tilde{\mathcal{L}}_{-\frac12}(N,\nu,r_j).$$

Combining [54, (4.36), (4.27) and the last equation of p. 502], for $r_j\ne\frac{i}{4}$ and $\tilde n>0$, since

$$L_{\frac32}\left(W_{\frac34\tilde n,\operatorname{Im}r}(4\pi\tilde n y)e(\tilde n x)\right)=-(\tfrac{1}{16}+r^2)W_{-\frac{\tilde n}{4},\operatorname{Im}r}(4\pi\tilde n y)e(\tilde n x),$$

the Fourier coefficient $\rho_j(n)$ of $v_j$ satisfies

$$\rho_j(n)=-(\tfrac{1}{16}+r^2)^{\frac12}\rho_j'(n)\qquad\text{for } r_j\ne\tfrac{i}{4},\ \tilde n>0, \tag{5.48}$$

and then

$$|\rho_j(n)|\asymp|\rho_j'(n)|\quad\text{if }|r_j|\le1,\ \operatorname{Im}r_j\le\tfrac{\theta}{2}\qquad\text{and}\quad|\rho_j(n)|\asymp r|\rho_j'(n)|\quad\text{if }r_j\ge1, \tag{5.49}$$

where the bound $2\operatorname{Im}r_j\le\theta$ is from Proposition 4.7.

In the case $r_j=\frac{i}{4}$, (2.13) and (2.14) show that $\rho_j(n)=0$ and

$$\tau_j(m,n)=0\qquad\text{for }\tilde n>0,\ r_j=\tfrac{i}{4}. \tag{5.50}$$

Moreover, by [54, (4.48)], if $E_{\mathfrak a}(z,s)$ is the Eisenstein series defined in weight $-\frac12$, then

$$L_{\frac32}E_{\mathfrak a}'(z,\tfrac12+ir)=(\tfrac14-ir)E_{\mathfrak a}(z,s)\quad\text{and}\quad(\tfrac{1}{16}+r^2)^{\frac12}|\rho_{\mathfrak a}'(n,r)|=|\rho_{\mathfrak a}(n,r)|.$$

We also get

$$|\rho_{\mathfrak{a}}(n,r)| \asymp |\rho'_{\mathfrak{a}}(n,r)| \quad \text{if } r\in[-1,1] \qquad \text{and} \quad |\rho_{\mathfrak{a}}(n,r)| \asymp r|\rho'_{\mathfrak{a}}(n,r)| \quad \text{if } |r|\geq 1. \tag{5.51}$$

We have the following proposition:

**Proposition 5.13.** *With the same setting as Theorem 1.7 for $k=-\frac12$, when $2x\geq A_u(m,n)^2$, we have*

$$\sum_{\substack{x<c\leq 2x\\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac{\theta}{2}]} (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1} \ll \left(x^{\frac16}+A_u(m,n)\right)(\tilde m\tilde n x)^{\varepsilon},$$

Note that here $\tau_j(m,n)$ is defined in weight $-\frac12$, i.e.

$$\tau_j(m,n) = 2i^{-\frac12}\overline{\rho_j(m)}\rho_j(n)\pi^{1-2s_j}(4\tilde m\tilde n)^{1-s_j}\frac{\Gamma(s_j-\frac14)\Gamma(2s_j-1)}{\Gamma(s_j+\frac14)}$$

where $\rho_j(n)$ is from (5.49) as the Fourier coefficient of $v_j\in\tilde{\mathcal{L}}_{-\frac12}(N,\nu,r_j)$.

The proof that Proposition 5.13 implies Theorem 1.7 in the case $k=-\frac12$ is the same as the case of weight $\frac12$ before. This is because $\tau_j(m,n)=0$ for $r_j=\frac{i}{4}$ (5.50) and because (5.32), (5.33) and (5.34) still hold for $r_j\in i(0,\frac{\theta}{2}]$ (the process only involves estimates on $\rho_j(n)$ with some applications of Proposition 4.15 in weight $-\frac12$). In the rest of this subsection we prove Proposition 5.13.

First we show that the main terms corresponding to $r_j=it_j\in i(0,\frac{\theta}{2}]$ are the same when we shift the weight between $-\frac12$ and $\frac32$. Recall $s_j=\frac12+t_j$. Let $\tau'_j(m,n)$ denote the corresponding coefficients for $x^{2s_j-1}$ in weight $\frac32$:

$$\tau'_j(m,n) = 2e^{\frac{3\pi i}{4}}\overline{\rho'_j(m)}\rho'_j(n)\pi^{-2t_j}(4\tilde m\tilde n)^{\frac12-t_j}\frac{\Gamma(\frac54+t_j)\Gamma(2t_j)}{\Gamma(t_j-\frac14)},$$

where $\rho'_j(n)$ is defined at the beginning of this subsection.

We claim that

$$\tau'_j(m,n)=\tau_j(m,n), \quad \text{for } \tilde m,\tilde n>0 \text{ and } r_j\in i(0,\tfrac14]. \tag{5.52}$$

When $r_j=\frac{i}{4}$, this is true because both of them equal to zero by (5.50) and $\Gamma(0)=\infty$. When $r_j\in i(0,\frac{\theta}{2}]$,

$$\begin{aligned}\tau_j(m,n) &= 2e^{-\frac{\pi i}{4}}\overline{\rho_j(m)}\rho_j(n)\pi^{-2t_j}(4\tilde m\tilde n)^{\frac12-t_j}\frac{\Gamma(\frac14+t_j)\Gamma(2t_j)}{\Gamma(\frac34+t_j)}\\ &= -2e^{\frac{3\pi i}{4}}\left(\frac{1}{16}-t_j^2\right)\overline{\rho'_j(m)}\rho'_j(n)\pi^{-2t_j}(4\tilde m\tilde n)^{\frac12-t_j}\frac{\Gamma(\frac54+t_j)/(\frac14+t_j)}{(-\frac14+t_j)\Gamma(-\frac14+t_j)}\Gamma(2t_j)\\ &= \tau'_j(m,n).\end{aligned}$$

Recall that the definition on $\widehat\phi$ (5.2) is for weight $k\geq 0$ and here we use $\widehat\phi$ for weight $\frac32$. We derive

$$4\sqrt{\tilde m\tilde n}\,\frac{\overline{\rho'_j(m)}\rho'_j(n)}{\operatorname{ch}\pi r_j}\widehat\phi(r_j) = (2^{2s_j-1}-1)\tau'_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}+O\left(A_u(m,n)(\tilde m\tilde n)^{\varepsilon}\right). \tag{5.53}$$

by the same process as we derive (4.49) above. Since $\tau'_j(m,n)=0$ when $r_j=\frac{i}{4}$, we have $2t_j\leq\theta<\delta$ (with

$\theta = \frac{7}{64}$ (2.15) and $\delta = \frac{1}{3}$ chosen later) by Proposition 4.7 and still get

$$\begin{aligned}
&\left| \sum_{\substack{x<c\le 2x\\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac{\theta}{2}]} (2^{2s_j-1}-1)\tau_j'(m,n)\frac{x^{2s_j-1}}{2s_j-1} \right| \\
&\le \left| \sum_{\substack{x<c\le 2x\\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{N|c>0} \frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right) \right| + O\left(A_u(m,n)(\tilde m\tilde n)^{\varepsilon}\right) \\
&\quad + \left| \sum_{N|c>0} \frac{S(m,n,c,\nu)}{c}\phi\left(\frac{a}{c}\right) - 4\sqrt{\tilde m\tilde n} \sum_{r_j\in i(0,\frac{\theta}{2}]} \frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j) \right| \\
&=: S_3 + O\left(A_u(m,n)(\tilde m\tilde n)^{\varepsilon}\right) + S_4.
\end{aligned} \tag{5.54}$$

The first sum $S_3$ above can be estimated similarly by condition (2) of Definition 1.6 as

$$S_3 \le \sum_{\substack{x-T\le c\le x\\ 2x\le c\le 2x+2T\\ N|c}} \frac{|S(m,n,c,\nu)|}{c} \ll_{N,\nu,\delta,\varepsilon} x^{\frac{1}{2}-\delta}(\tilde m\tilde n x)^{\varepsilon}. \tag{5.55}$$

By Theorem 5.1,

$$S_4 \ll |\mathcal{U}_{\frac{3}{2}}| + \left| \sqrt{\tilde m\tilde n}\sum_{r_j\ge 0} \frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j) + \sqrt{\tilde m\tilde n}\sum_{\text{singular } \mathfrak{a}} \int_{-\infty}^{\infty} \overline{\rho_{\mathfrak{a}}'(m,r)}\rho_{\mathfrak{a}}'(n,r)\frac{\widehat{\phi}(r)}{\operatorname{ch}\pi r}dr \right|.$$

The bound for $\mathcal{U}_{\frac{3}{2}}$ is done in (5.40). Estimates for the remaining part of $S_4$ follow from the same process as §5.3.1.2 in the case of weight $\frac{1}{2}$, taking (5.49) and (5.51) into account. For the same reason as the beginning of §5.3.1.2, we just record the bounds with respect to the discrete spectrum here.

For $r \in [0,1)$, we apply Proposition 4.15, (5.49) and (5.7) to get

$$\sqrt{\tilde m\tilde n}\sum_{r\in[0,1)} \left| \frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j) \right| \ll \sqrt{\tilde m\tilde n}\sum_{r\in[0,1)} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j) \right| \ll A(m,n)(\tilde m\tilde n x)^{\varepsilon}. \tag{5.56}$$

For $r \in [1, \frac{a}{x})$, we apply Proposition 4.15, $\rho_j'(n) \ll r_j^{-1}|\rho_j(n)|$ from (5.49), and $\widehat{\phi}(r) \ll 1$ from (5.25). Since

$$s(R) := \sqrt{\tilde m\tilde n}\sum_{r\in[1,R]} \left| \frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j} \right| \ll A(m,n)R^{\frac{7}{2}}(\tilde m\tilde n)^{\varepsilon} \tag{5.57}$$

by Cauchy-Schwarz, with the help of (5.29) we have

$$\begin{aligned}\sqrt{\tilde{m}\tilde{n}}\sum_{r_j\in[1,\frac{a}{x})}\left|\frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right| &\ll \sqrt{\tilde{m}\tilde{n}}\sum_{r_j\in[1,\frac{a}{x})}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}r_j^{-2}\right| \\ &\ll r^{-2}s(r)\Big|_{r=1}^{\frac{a}{x}}+\int_1^{\frac{a}{x}}s(r)r^{-3}dr \\ &\ll A(m,n)\left(\frac{a}{x}\right)^{\frac{3}{2}}(\tilde{m}\tilde{n}x)^{\varepsilon} \\ &\ll A_u(m,n)(\tilde{m}\tilde{n}x)^{\varepsilon}.\end{aligned}\tag{5.58}$$

We still let

$$P(m,n)=2(\tilde{m}\tilde{n})^{\frac{1}{8}}A(m,n)^{-\frac{1}{2}}\geq 1$$

and divide $r\geq\max(\frac{a}{x},1)$ into two parts: $\max\left(\frac{a}{x},1\right)\leq r<P(m,n)$ and $r\geq\max\left(\frac{a}{x},1,P(m,n)\right)$. In the first range, we apply Proposition 4.15, (5.49) and $\widehat{\phi}(r)\ll 1$ from (5.25) to get

$$\sqrt{\tilde{m}\tilde{n}}\sum_{\max(\frac{a}{x},1)\leq r_j<P(m,n)}\left|\frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\ll A_u(m,n)(\tilde{m}\tilde{n}x)^{\varepsilon}\tag{5.59}$$

by partial summation similar as (5.58). We divide the second range into dyadic intervals $C\leq r_j<2C$ and apply Proposition 4.14, (5.49) and $\widehat{\phi}(r)\ll\min(1,\frac{x}{rT})$ from (5.25):

$$\begin{aligned}\sqrt{\tilde{m}\tilde{n}}\sum_{C\leq r_j<2C}\left|\frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right| &\ll \sqrt{\tilde{m}\tilde{n}}\sum_{C\leq r_j<2C}\left|\frac{\overline{\rho_j(m)}\rho_j(n)}{\operatorname{ch}\pi r_j}r_j^{-2}\widehat{\phi}(r_j)\right| \\ \ll \min\left(1,\frac{x}{CT}\right)C^{-2}\left(C^{\frac{5}{2}}+(\tilde{m}^{\frac{1}{4}}+\tilde{n}^{\frac{1}{4}})C^{\frac{3}{2}}+(\tilde{m}\tilde{n})^{\frac{1}{4}}C^{\frac{1}{2}}\right)(\tilde{m}\tilde{n}x)^{\varepsilon} \\ \ll \left(\min\left(C^{\frac{1}{2}},C^{-\frac{1}{2}}\frac{x}{T}\right)+(\tilde{m}^{\frac{1}{4}}+\tilde{n}^{\frac{1}{4}})C^{-\frac{1}{2}}+(\tilde{m}\tilde{n})^{\frac{1}{4}}C^{-\frac{3}{2}}\right)(\tilde{m}\tilde{n}x)^{\varepsilon}.\end{aligned}\tag{5.60}$$

Summing up from (5.60) similar as we did after (4.58) and recalling $T\asymp x^{1-\delta}$ in Setting 4.8, we have

$$\begin{aligned}&\sqrt{\tilde{m}\tilde{n}}\sum_{r_j\geq\max(\frac{a}{x},1,P(m,n))}\left|\frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right| \\ &\ll\left(\left(\frac{x}{T}\right)^{\frac{1}{2}}+(\tilde{m}+\tilde{n})^{\frac{1}{4}}(\tilde{m}\tilde{n})^{-\frac{1}{16}}A(m,n)^{\frac{1}{4}}+(\tilde{m}\tilde{n})^{\frac{1}{16}}A(m,n)^{\frac{3}{4}}\right)(\tilde{m}\tilde{n}x)^{\varepsilon} \\ &\ll\left(x^{\frac{\delta}{2}}+(\tilde{m}\tilde{n})^{\frac{3}{16}}A(m,n)^{\frac{1}{4}}\right)(\tilde{m}\tilde{n}x)^{\varepsilon}.\end{aligned}\tag{5.61}$$

From (5.59) and (5.61) we have

$$\sqrt{\tilde{m}\tilde{n}}\sum_{r_j\geq\max(\frac{a}{x},1)}\left|\frac{\overline{\rho_j'(m)}\rho_j'(n)}{\operatorname{ch}\pi r_j}\widehat{\phi}(r_j)\right|\ll\left(x^{\frac{\delta}{2}}+A_u(m,n)\right)(\tilde{m}\tilde{n}x)^{\varepsilon}.\tag{5.62}$$

Combining (5.54), (5.55), (5.40), (5.56), (5.58), and (5.62), we get

$$\sum_{\substack{x<c\leq 2x\\ N|c}} \frac{S(m,n,c,\nu)}{c} - \sum_{r_j\in i(0,\frac{1}{4}]} (2^{2s_j-1}-1)\tau_j(m,n)\frac{x^{2s_j-1}}{2s_j-1}$$
$$\ll \left(x^{\frac{1}{2}-\delta}+x^{\frac{\delta}{2}}+A_u(m,n)\right)(\tilde{m}\tilde{n}x)^{\varepsilon}.$$

Proposition 5.13 follows by choosing $\delta=\frac{1}{3}$ and we finish the proof of Theorem 1.7.

*Proof of Theorem 1.9.* The proof follows from the same process as [14, §9.2]. Note that we need to restrict $\sum_{r_j=\frac{i}{4}}\tau_j(m,n)=0$ when $\tilde{m}>0$, $\tilde{n}>0$ and $k=\frac{1}{2}$ (and the conjugate case $\tilde{m}<0$, $\tilde{n}<0$ and $k=-\frac{1}{2}$ by (1.13)), otherwise the sum may not converge. □

# Chapter 6

# Partitions of rank modulo 3

In this section we prove Theorem 1.11, with help from Theorem 1.7. The idea is essentially the same as [7], [21]: we construct a particular weight $\frac{1}{2}$ Maass-Poincaré series, whose holomorphic part of their value at $s = \frac{3}{4}$ is the rank generating function $q^{-\frac{1}{24}}\mathcal{R}(w;q)$. These Maass-Poincaré series are convergent uniformly and absolutely when $\operatorname{Re} s > 1$ by definition. To prove their convergence at $s = \frac{3}{4}$, we need the uniform bound in Theorem 1.7.

## 6.1 Proof of Theorem 1.11, main line

Now we use the theorems in Section 1 to prove Theorem 1.11. We follow the outline of [7] and the idea is that $q^{-\frac{1}{24}}\mathcal{R}(w,q)$ is the holomorphic part of a Poincaré series whose Fourier coefficients can be explicitly calculated.

Recall the notations in Section 2.3. Let $\nu$ be an admissible multiplier system on $\Gamma_0(N)$ where $\alpha_\nu > 0$ and $B$, $M$, $\nu'$ and $D$ be as in Definition 1.6. Let

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn} y,\ s-\frac{1}{2}}(|y|) \quad \text{and} \quad \varphi_{s,k}(z) := \mathcal{M}_s(4\pi y)e(x)$$

where $M_{\alpha,\beta}$ is the standard $M$-Whittaker function. One can check that $\varphi_{s,k}(z)$ is an eigenfunction of $\widetilde{\Delta}_k$ with eigenvalue $s(1-s) + \frac{k^2-2k}{4}$. We define the Maass-Poincaré series by

$$P_k(s,m,N;z) := \frac{1}{\Gamma(2-k)} \sum_{\gamma=\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma_0(N)} \overline{\nu}(\gamma)(cz+d)^{-k}\varphi_{s,k}(\tilde{m}\gamma z). \tag{6.1}$$

By [21, Lemma 3.1], when $\operatorname{Re} s > 1$, the above series is absolutely and uniformly convergent (in any compact subset) . As in [21, Theorem 3.2 & Remark (1)], we have the following theorem for $P_k$ (note that we have replaced their $2-k$ by $k$). Recall that $H_k$ denotes the space of harmonic Maass forms of weight $k$ in Definition 2.3.

**Theorem 6.1.** *With the notation above, when $k \le -\frac{1}{2}$ is half-integral and $\tilde{m} < 0$, we have*

$$P_k(1-\tfrac{k}{2}, m, N; Bz) \in H_k(\Gamma_0(M), (\tfrac{|D|}{\cdot})\nu_\theta^{2k})$$

*and*

$$P_k\left(1-\tfrac{k}{2},m,N;z\right)=\frac{1-k}{\Gamma(2-k)}\big(\Gamma(1-k)-\Gamma(1-k,4\pi|\tilde{m}|y)\big)q^{\tilde{m}}$$
$$+\sum_{\tilde{n}>0}\beta(n)q^{\tilde{n}}+\sum_{\tilde{n}<0}\beta'(n)\Gamma\left(1-k,4\pi|\tilde{n}|y\right)q^{\tilde{n}},$$

*where*

$$\beta(n)=i^{-k}2\pi\left|\frac{\tilde{m}}{\tilde{n}}\right|^{\frac{1-k}{2}}\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}I_{1-k}\left(\frac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right)$$

*and*

$$\beta'(n)=\frac{i^{-k}2\pi}{\Gamma(1-k)}\left|\frac{\tilde{m}}{\tilde{n}}\right|^{\frac{1-k}{2}}\sum_{N|c>0}\frac{S(m,n,c,\nu)}{c}J_{1-k}\left(\frac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right).$$

*This theorem also holds when $k=\frac{1}{2}$, provided that we ensure the convergence of the formulas for $\beta(n)$ and $\beta'(n)$. We can guarantee the convergence of these formulas for any admissible multiplier $\nu$ satisfying $\alpha_\nu>0$.*

We prove Theorem 1.11 assuming Theorem 6.1 in this section and prove Theorem 6.1 in the next section. Define the theta function as

$$\theta(z;h,N):=\sum_{n\equiv h \pmod N} nq^{\frac{n^2}{24}}.$$

It is well known that the above theta functions are holomorphic cusp forms of weight $\frac{3}{2}$ whose transformation formulas can be computed via [55]. Moreover, Bringmann and Ono [8] showed that some period integral of a linear combination of such theta functions can be added as a non-holomorphic part to $q^{-\frac{1}{24}}\mathcal{R}(w;q)$ to get a harmonic Maass form. We call that combination a shadow of $q^{-\frac{1}{24}}\mathcal{R}(w;q)$.

When $w\neq 1$ is a root of unity, by [56, Theorem 7.1] we know that $q^{-\frac{1}{24}}\mathcal{R}(w;q)$ is a mock modular form of weight $\frac{1}{2}$ with shadow proportional to

$$\left(w^{\frac{1}{2}}-w^{-\frac{1}{2}}\right)\sum_{n\in\mathbb{Z}}\left(\frac{12}{n}\right)nw^{\frac{n}{2}}q^{\frac{n^2}{24}}. \tag{6.2}$$

Hence the shadow of $q^{-\frac{1}{24}}\mathcal{R}(-1;q)$ is proportional to $\theta(z;1,6)$ as [7, Remark, p.251] and a computation shows that the shadow of $q^{-\frac{1}{24}}\mathcal{R}(e^{\frac{2\pi i}{3}};q)$ is proportional to $\theta(z;1,12)+\theta(z;5,12)$. Moreover, the differential operator $\xi_k$ maps a weight $k$ harmonic Maass form to its shadow.

We take our weight $\frac{1}{2}$ multiplier system $\nu=(\frac{d}{3})\overline{\nu_\eta}$ on $\Gamma_0(3)$ to define the Maass-Poincaré series $P_k(s,m,3;z)$ in (6.1). This multiplier is admissible with $B=24$ and $|D|=4$, i.e. the trivial Nebentypus. Denote

$$P(z):=P_{\frac{1}{2}}(\tfrac{3}{4},0,3;z),\quad \text{so } P(24z)\in H_{\frac{1}{2}}(576,\nu_\theta)$$

and write the Fourier expansion of $P(z)$ as in Theorem 6.1.

We define $M(z)$ to be the unique harmonic Maass form such that $q^{-\frac{1}{24}}\mathcal{R}(e^{\frac{2\pi i}{3}};q)$ is its holomorphic part. It follows that $M(24z)$ is a weight $\frac{1}{2}$ harmonic Maass form for some $\Gamma_0(M')$ and Nebentypus $\chi'$. If we can establish the equality $M(24z)=P(24z)$, then Theorem 1.11 is proved using Theorem 6.1. The rest of this section is devoted to proving this equality.

Decompose $P(24z)$ and $M(24z)$ into holomorphic and non-holomorphic parts as

$$P(24z) = P_h(24z) + P_{nh}(24z) = q^{-1} + \sum_{n=1}^{\infty} \beta(n) q^{24n-1} + P_{nh}(24z), \tag{6.3}$$

$$M(24z) = M_h(24z) + M_{nh}(24z) = q^{-1}\mathcal{R}(e^{\frac{2\pi i}{3}}; q^{24}) + M_{nh}(24z). \tag{6.4}$$

**Lemma 6.2.** *$M(z)$ is a weight $\frac{1}{2}$ harmonic Maass form for $(\Gamma_0(3), (\frac{d}{3})\overline{\nu_\eta})$.*

*Proof.* We begin by investigating the shadows. Recall that $S_k(N, \nu)$ is the space of holomorphic cusp forms on $\Gamma_0(N)$ with multiplier system $\nu$. By combining Lemma 2.4, Theorem 6.1 and the definition of $\xi_{\frac{1}{2}}$, the shadow of $P$ satisfies

$$P_{sha}(z) := \xi_{\frac{1}{2}}(P(z)) = \xi_{\frac{1}{2}}(P_{nh}(z)) \in S_{\frac{3}{2}}(3, (\tfrac{\cdot}{3})\nu_\eta). \tag{6.5}$$

Direct calculations using (1.19) yield $P_{sha}(3z) \in S_{\frac{3}{2}}(9, \nu_\eta^3)$.

On the other hand, since $\xi_{\frac{1}{2}}$ maps $M(z)$ to the shadow of $q^{-\frac{1}{24}}\mathcal{R}(e^{\frac{2\pi i}{3}}; q)$, we see that $\xi_{\frac{1}{2}}(M(z))$ is proportional to

$$M_{sha}(z) := \theta(z; 1, 12) + \theta(z; 5, 12) = \sum_{n=1}^{\infty} \chi_{-36}(n) n q^{\frac{n^2}{24}},$$

where $\chi_{-36}$ is the Dirichlet character modulo 12 induced by $(\frac{-4}{\cdot})$. One can check that $M_{sha}(3z) = \eta(z)^3 - (\eta^3|U_3V_3)(z)$ where for $f = \sum_{n=1}^{\infty} a_f(n) q^{\frac{n}{8}}$,

$$(f|U_3)(z) := \sum_{n=1}^{\infty} a_f(3n) q^{\frac{n}{8}} = \frac{1}{3}\sum_{u=0}^{2} f\left(\frac{z+8u}{3}\right) \quad \text{and} \quad (f|V_3)(z) := f(3z).$$

Clearly $\eta^3 \in S_{\frac{3}{2}}(1, \nu_\eta^3)$. With some tedious matrix calculation, we observe

$$(\eta^3|U_3V_3)(z) = \frac{1}{3}\sum_{u=0}^{2} \eta^3\left(z + \frac{8u}{3}\right) \in S_{\frac{3}{2}}(9, \nu_\eta^3).$$

Hence $M_{sha}(3z) \in S_{\frac{3}{2}}(9, \nu_\eta^3)$, so $P_{sha}(3z)$ and $M_{sha}(3z)$ are in the same space.

Next we prove that $P_{sha}(3z)$ and $M_{sha}(3z)$ are proportional. One can check that

$$\eta(3z) \in S_{\frac{1}{2}}(9, (\tfrac{\cdot}{3})\nu_\eta^3) \quad \text{and} \quad f(z) \in S_{\frac{3}{2}}(9, \nu_\eta^3) \implies f(z)\eta(3z)^7 \in S_5(9, (\tfrac{\cdot}{3})).$$

Here $S_5(9, (\frac{\cdot}{3}))$ is a two-dimensional space spanned by $q - 2q^4 + O(q^6)$ and $q^2 + q^3 + O(q^4)$. Since both $P_{sha}(3z)$ and $M_{sha}(3z)$ have Fourier expansion

$$\sum_{n \equiv 1 \pmod{24}} a_1(n) q^{\frac{n}{8}}, \qquad \text{and} \quad \eta(3z)^7 = \sum_{n \equiv 7 \pmod{24}} a_2(n) q^{\frac{n}{8}},$$

the Fourier expansion of $P_{sha}(3z)\eta(3z)^7$ and $M_{sha}(3z)\eta(3z)^7$ both start with $Cq + O(q^4)$ for some non-zero constant $C$ (which might be different). We get

$$P_{sha}(3z)\eta(3z)^7 = cM_{sha}(3z)\eta(3z)^7 \quad \Rightarrow \quad P_{sha}(z) = cM_{sha}(z) \ \text{ and } \ P_{nh}(z) = cM_{nh}(z)$$

for some constant $c$.

From (6.5) we conclude that $M_{sha}(z) \in S_{\frac{3}{2}}(3, (\frac{\cdot}{3})\nu_\eta)$. By [8, Theorem 1.2] we know that $M(72z)$ is a harmonic Maass form on $\Gamma_1(1728)$ and by [8, Theorem 3.4], $M(z)$ is an entry of a vector-valued harmonic Maass form on $\Gamma_0(1)$. To prove Lemma 6.2, it suffices to check the transformation law on $\Gamma_0(3)$ and we do not need to check for the growth rate at the cusps. It is known that $\Gamma_0(3)$ can be generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and

$$\left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & -1 \\ 3 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -1/3 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right). \tag{6.6}$$

The transformation law of $M_h(z) = q^{-\frac{1}{24}}\gamma(q)$ as a sixth-order mock theta function can be found in [57, (4.3), (5.5), p. 122]. Combining $M_h$ and $M_{nh}$ and carefully comparing the notation of Mordell integrals between [57, p. 121]

$$J(\alpha) := \int_0^\infty \frac{e^{-\alpha x^2}}{\operatorname{ch} \alpha x} dx$$

and [8, (2.5), Theorem 2.3, (3.2), and Lemma 3.2]

$$J(\tfrac{1}{3};\alpha) := \int_0^\infty e^{-\frac{3}{2}\alpha x^2}\, \frac{\operatorname{ch}\alpha x + 1}{\operatorname{ch}(3\alpha x/2)} dx,$$

we check that, under the transform of generators of $\Gamma_0(3)$ decomposing as in (6.6),

$$M(\left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right) z) = e\left(\tfrac{5}{12}\right)(2-3z)^{\frac{1}{2}} M(z), \quad \text{where } (\tfrac{2}{3})\overline{\nu_\eta}\left(\begin{smallmatrix} -1 & 1 \\ -3 & 2 \end{smallmatrix}\right) = e\left(\tfrac{5}{12}\right)$$

with the help of (1.19).

□

Next we show that the principal part of $M(z)$ at the cusp 0 of $\Gamma_0(3)$ is constant. We can take the scaling matrix $\sigma_0 = \left(\begin{smallmatrix} & -1 \\ 3 & \end{smallmatrix}\right)$. With [8, Theorem 2.3] we can check the image of the holomorphic part $M_h(z) = \sin\frac{\pi}{3}\mathcal{N}(\frac{1}{3};q)$ (in their notation) under the slash operator $|_{\frac{1}{2}}\sigma_0$. The result $\mathcal{M}(\frac{1}{3};3z)$ has principal part 0 and the Mordell integral $\sqrt{z}J(\frac{1}{3};-6\pi i z)$ is bounded when $\operatorname{Im} z \to \infty$.

Since the principal part of the Poincaré series $P(z)$ is non-constant only at $\infty$, by (6.3), the principal part of $E(z) := P(z) - M(z)$ is constant at both cusps of $\Gamma_0(3)$. Then $P_{nh} = M_{nh}$ by [21, Lemma 2.3] and $E(z) = P_h(z) - M_h(z)$ is in fact a holomorphic modular form whose Fourier coefficients are supported on $n - \frac{1}{24}$ for $n \geq 1$.

According to the Serre-Stark basis theorem [32, Theorem A], the space $M_{\frac{1}{2}}(576, \nu_\theta)$ consists of theta functions whose Fourier coefficients are zero except those for exponents $t\ell^2$ where $t|576$ and $\ell \in \mathbb{Z}$. However, $E(24z)$ is in $M_{\frac{1}{2}}(576, \nu_\theta)$ and has Fourier expansion supported on exponents of the form $24n-1$. Therefore, $E(z) = 0$.

It follows that $q^{-\frac{1}{24}}\mathcal{R}(e^{\frac{2\pi i}{3}};q) = P_h(z)$ is holomorphic part of $P_{\frac{1}{2}}(\frac{3}{4}, 0, 3; z)$ whose Fourier coefficient is shown in Theorem 6.1. Note that we are in the special case $k = \frac{1}{2}$ and $\tilde{m} = \tilde{0} = -\frac{1}{24} < 0$ (stated at the end of Theorem 6.1). This finishes the proof of Theorem 1.11.

*Remark.* The exact formula of $A(\frac{1}{2};n)$ in [7], which can be rewritten as (1.34), can also be deduced from a similar process as our proof here using Theorem 6.1.

## 6.2 Proof of Theorem 6.1

The only thing left is to prove the Fourier expansion of Theorem 6.1. Recall the notations in §2.3. Let $M_{\alpha,\beta}$ and $W_{\alpha,\beta}$ denote the $M$- and $W$-Whittaker functions, respectively. For $y>0$, by [31, (13.18.4)],

$$\begin{aligned}\mathcal{M}_{1-\frac{k}{2}}(-y)&=y^{-\frac{k}{2}}M_{-\frac{k}{2},\,\frac12-\frac{k}{2}}(y)\\&=(1-k)(\Gamma(1-k)-\Gamma(1-k,y))e^{\frac{y}{2}},\end{aligned}\tag{6.7}$$

and by [31, (13.18.2)],

$$W_{-\frac{k}{2},\,\frac12-\frac{k}{2}}(y)=y^{\frac{k}{2}}e^{\frac{y}{2}}\Gamma(1-k,y),\quad W_{\frac{k}{2},\,\frac12-\frac{k}{2}}(y)=y^{\frac{k}{2}}e^{-\frac{y}{2}}.\tag{6.8}$$

The contribution to $P_k(1-\frac{k}{2},m,N;z)$ from $c=0$ in (6.1) equals

$$\frac{1}{\Gamma(2-k)}\varphi_{1-\frac{k}{2},\,k}(\tilde{m}z)=\frac{1-k}{\Gamma(2-k)}(\Gamma(1-k)-\Gamma(1-k,4\pi|\tilde{m}|y))e^{2\pi\tilde{m}z}.$$

The contribution to $P_k(1-\frac{k}{2},m,N;z)$ from some $c>0$ equals

$$\begin{aligned}&\frac{1}{\Gamma(2-k)}\sum_{\ell\in\mathbb{Z}}\sum_{\substack{d(c)^*\\0<a<c,\ ad\equiv1(c)}}\overline{\nu}\left(\begin{smallmatrix}a&*\\c&d+\ell c\end{smallmatrix}\right)(cz+d+\ell c)^{-k}\\&\qquad\cdot\mathcal{M}_{1-\frac{k}{2}}\left(\frac{4\pi\tilde{m}y}{|cz+d+\ell c|^2}\right)e\left(\frac{\tilde{m}a}{c}-\operatorname{Re}\left(\frac{\tilde{m}}{c(cz+d+\ell c)}\right)\right)\\&=\frac{1}{\Gamma(2-k)}c^{-k}\sum_{d(c)^*}\overline{\nu}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)e\left(\frac{\tilde{m}a}{c}\right)\sum_{\ell\in\mathbb{Z}}e(\ell\alpha_\nu)\left(z+\frac{d}{c}+\ell\right)^{-k}\\&\qquad\cdot\mathcal{M}_{1-\frac{k}{2}}\left(\frac{4\pi\tilde{m}y}{c^2|z+\frac{d}{c}+\ell|^2}\right)e\left(-\frac{\tilde{m}}{c^2}\operatorname{Re}\left(\frac{1}{z+\frac{d}{c}+\ell}\right)\right),\end{aligned}\tag{6.9}$$

where we used (1.9): $\nu\left(\begin{smallmatrix}a&b+\ell a\\c&d+\ell c\end{smallmatrix}\right)=\nu\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\nu\left(\begin{smallmatrix}1&\ell\\0&1\end{smallmatrix}\right)=\nu\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)e(-\ell\alpha_\nu)$ for all $\ell\in\mathbb{Z}$. Let

$$f(z):=\sum_{\ell\in\mathbb{Z}}\frac{e(\ell\alpha_\nu)}{(z+\ell)^k}\,\mathcal{M}_{1-\frac{k}{2}}\left(\frac{4\pi\tilde{m}y}{c^2|z+\ell|^2}\right)e\left(-\frac{\tilde{m}}{c^2}\operatorname{Re}\left(\frac{1}{z+\ell}\right)\right).$$

Then $f(z)e(\alpha_\nu x)$ has period 1 and $f$ has Fourier expansion

$$f(z)=\sum_{n\in\mathbb{Z}}a_y(n)e(\tilde{n}x),\quad f\left(z+\frac{d}{c}\right)=\sum_{n\in\mathbb{Z}}a_y(n)e\left(\frac{\tilde{n}d}{c}\right)e(\tilde{n}x),\tag{6.10}$$

where by (6.7)

$$\begin{aligned}a_y(n)&=\int_{\mathbb{R}}z^{-k}\mathcal{M}_{1-\frac{k}{2}}\left(\frac{4\pi\tilde{m}y}{c^2|z|^2}\right)e\left(-\frac{\tilde{m}x}{c^2|z|^2}-nx+\alpha_\nu x\right)dx\\&=\frac{c^k}{|4\pi\tilde{m}y|^{\frac{k}{2}}}\int_{\mathbb{R}}\left(\frac{x-iy}{x+iy}\right)^{\frac{k}{2}}M_{-\frac{k}{2},\frac12-\frac{k}{2}}\left(\frac{4\pi|\tilde{m}|y}{c^2|z|^2}\right)e\left(-\frac{\tilde{m}x}{c^2|z|^2}-\tilde{n}x\right)dx.\end{aligned}$$

By substituting $x = yu$,

$$a_y(n) = \frac{i^{-k} y c^k}{|4\pi\tilde{m}y|^{\frac{k}{2}}} \int_{\mathbb{R}} \left(\frac{1+iu}{1-iu}\right)^{\frac{k}{2}} M_{-\frac{k}{2},\frac{1}{2}-\frac{k}{2}} \left(\frac{4\pi|\tilde{m}|}{c^2 y(u^2+1)}\right) e\left(\frac{|\tilde{m}|u}{c^2 y(u^2+1)} - \tilde{n}yu\right) du.$$

This integral is evaluated by [58, p.32-33]. We get

$$a_y(n) = \frac{i^{-k} c^k \Gamma(2-k)}{|4\pi\tilde{m}y|^{\frac{k}{2}} c} \cdot \begin{cases} \dfrac{2\pi\sqrt{|\tilde{m}/\tilde{n}|}}{\Gamma(1-k)} W_{-\frac{k}{2},1-\frac{k}{2}}(4\pi|\tilde{n}|y) J_{1-k}\left(\dfrac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right), & \tilde{n} < 0; \\ \dfrac{4\pi^{2-\frac{k}{2}} |\tilde{m}|^{1-\frac{k}{2}} c^{k-1} y^{\frac{k}{2}}}{(1-k)\Gamma(1-k)}, & \tilde{n} = 0; \\ 2\pi\sqrt{|\tilde{m}/\tilde{n}|}\, W_{\frac{k}{2},1-\frac{k}{2}}(4\pi\tilde{n}y) I_{1-k}\left(\dfrac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right), & \tilde{n} > 0. \end{cases}$$

Applying (6.8), substituting (6.10) in (6.9), interchanging the finite sum on $d$ and sum on $n$, and summing over $N|c > 0$ we get

$$P_k(1-\tfrac{k}{2}, m, N; z) = \frac{1-k}{\Gamma(2-k)}(\Gamma(1-k) - \Gamma(1-k, 4\pi|\tilde{m}|y)) e^{2\pi\tilde{m}z} + \sum_{n\in\mathbb{Z}} e^{2\pi i\tilde{n}z}$$
$$\cdot \begin{cases} \dfrac{i^{-k} 2\pi\Gamma(1-k, 4\pi|\tilde{n}|y)}{\Gamma(1-k)} \left|\dfrac{\tilde{m}}{\tilde{n}}\right|^{\frac{1-k}{2}} \displaystyle\sum_{N|c>0} \dfrac{S(m,n,c,\nu)}{c} J_{1-k}\left(\dfrac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right), & \tilde{n} < 0; \\ \dfrac{i^{-k}(2\pi)^{2-k}|\tilde{m}|^{1-k}}{\Gamma(2-k)} \displaystyle\sum_{N|c>0} \dfrac{S(m,0,c,\nu)}{c^{2-k}} & \tilde{n} = 0; \\ i^{-k} 2\pi \left|\dfrac{\tilde{m}}{\tilde{n}}\right|^{\frac{1-k}{2}} \displaystyle\sum_{N|c>0} \dfrac{S(m,n,c,\nu)}{c} I_{1-k}\left(\dfrac{4\pi\sqrt{|\tilde{m}\tilde{n}|}}{c}\right), & \tilde{n} > 0. \end{cases}$$

Since we assumed $\alpha_\nu > 0$, we do not have the term for $\tilde{n} = 0$.

It remains to prove the convergence of Fourier coefficients when $k = \frac{1}{2}$ and $\alpha_\nu > 0$. Since $\tilde{m} < 0$, whenever $\tilde{n} > 0$ or $< 0$, the convergence follows from Theorem 1.9. Readers may notice that the original proof in [14, Section 10] involves Cauchy's convergence when $\tilde{n} < 0$. As we already proved Theorem 1.9 in both the mixed- and same-sign case, we have finished the proof.

## 6.3 Asymptotics for ranks of partitions modulo 1,2,3

In this section we collect some results on the asymptotics to the exact formulas for ranks of partitions modulo $p \le 3$. Note that the exact formulas for $p \ge 5$ become different and we will discuss in the next chapter. Recall the definition of $A_c(n)$ in (1.26). The first asymptotic for $p(n)$ is given by Hardy and Ramanujan [4]: let $\tilde{n} = n - \frac{1}{24}$, then

$$p(n) \sim \frac{1}{2\pi\sqrt{2}} \sum_{c \le \alpha\sqrt{n}} A_c(n) c^{\frac{1}{2}} \frac{d}{dn} \left(\frac{\exp\left(\pi\sqrt{\frac{2\tilde{n}}{3}}/c\right)}{\sqrt{\tilde{n}}}\right). \tag{6.11}$$

Later in 1938, Rademacher proved the famous exact formula, by showing that [5, (5.9)]

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{c=1}^{N} A_c(n) c^{\frac{1}{2}} \frac{d}{dn} \left( \frac{\sinh\left(\pi\sqrt{\frac{2\tilde{n}}{3}}/c\right)}{\sqrt{\tilde{n}}} \right) + O(e^{2\pi n/N^2} N^{-\frac{1}{2}}) \tag{6.12}$$

and letting $N \to \infty$.

Let $R_1(n;x)$ be the tail sum:

$$R_1(n;x) := \frac{1}{\pi\sqrt{2}} \sum_{c \geq x} A_c(n) c^{\frac{1}{2}} \frac{d}{dn} \left( \frac{\sinh\left(\pi\sqrt{\frac{2\tilde{n}}{3}}/c\right)}{\sqrt{\tilde{n}}} \right).$$

This sum is convergent by Rademacher's exact formula. Rademacher's result showed $R_1(n, \alpha\sqrt{n}) \ll n^{-\frac{1}{4}}$, which was later improved to $R_1(n, \alpha\sqrt{n}) \ll n^{-\frac{1}{2}} \log n$ by Lehmer [59]. The recent work by Ahlgren and Andersen [11, Theorem 1.1] proved $R_1(n, \alpha\sqrt{n}) \ll n^{-\frac{1}{2}-\frac{1}{168}+\varepsilon}$ and optimized the bound to $R_1(n, \alpha n^{\frac{1}{2}+\frac{5}{252}}) \ll n^{-\frac{1}{2}-\frac{1}{28}+\varepsilon}$. The best estimate known today is by Andersen and Wu [13]:

$$R_1(n, \alpha\sqrt{n}) \ll n^{-\frac{1}{2}+\varepsilon} t^{-\frac{1}{36}} w^{-\frac{1}{6}}, \quad \text{where } 24n-1 = tw^2 \text{ and } t \text{ is square-free.}$$

Recall $\mathcal{R}(w;q)$ defined in (1.31) and $A(\frac{\ell}{u};n)$ as its Fourier coefficient when $w = \zeta_u^\ell = e(\ell/u)$. As $\mathcal{R}(-1;q)$ is one of Ramanujan's famous third order mock theta functions, Ramanujan claimed a similar asymptotic for $A(\frac{1}{2};n) = N(0,2;n) - N(1,2;n)$, which was proved by Dragonette in 1952 [9]:

$$A\left(\frac{1}{2};n\right) = (-1)^{n-1} \frac{\exp(\pi\sqrt{\tilde{n}/6})}{2\sqrt{\tilde{n}}} + O\left( \frac{\exp(\frac{\pi}{2}\sqrt{\tilde{n}/6})}{2\sqrt{\tilde{n}}} \right).$$

Dragonette concluded a even stronger result:

$$A\left(\frac{1}{2};n\right) = \sum_{c=1}^{\sqrt{n}} \frac{\lambda(c)\exp(\frac{\pi}{c}\sqrt{\tilde{n}/6})}{\sqrt{c}\sqrt{\tilde{n}}} + O(n^{\frac{1}{2}} \log n),$$

where $\lambda(1) = (-1)^{n-1}/2$ and

$$\lambda(c) = \frac{1}{2} \sum_{\substack{(h,2c)=1 \\ -c<h<c}} e\left(\frac{hn}{2c}\right) \varepsilon_{h,c}$$

for $h$ odd, $hh'' \equiv -1 \pmod{2c}$ and

$$\varepsilon_{h,c} = (-1)^{c+\frac{h^2-1}{8}} e\left( \frac{(2c^2+1)h''(h^2-1)/2 - h(c^2-1)}{24c} \right) \cdot \frac{1}{2c^2} \sum_{\mu=0}^{2c-1} e\left( \frac{(h+c)\mu^2}{2c} \right).$$

Andrews [10] improved Dragonette's result as

$$A\left(\frac{1}{2};n\right) = \sum_{c=1}^{\sqrt{n}} \frac{\lambda(c)\exp(\frac{\pi}{c}\sqrt{\tilde{n}/6})}{\sqrt{c}\sqrt{\tilde{n}}} + O_\varepsilon(n^\varepsilon) \tag{6.13}$$

for any $\varepsilon > 0$. The exact formula was finally proved by Bringmann and Ono [7], as we have stated in (1.33).

If we let $R_2(n,x)$ be the tail sum

$$R_2(n,x) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k\geq x} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}(n - \frac{k(1+(-1)^k)}{4})}{k} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right),$$

then Dragonette's result implies $R_2(n,\sqrt{n}) \ll n^{\frac{1}{2}} \log n$ and Andrews' result implies $R_2(n,\sqrt{n}) \ll_\varepsilon n^\varepsilon$. The work by Ahlgren and Dunn [12] improved the bound to

$$R_2(n,\alpha\sqrt{n}) \ll_{\alpha,\varepsilon} n^{-\frac{1}{147}+\varepsilon}$$

when $24n-1$ is square-free. The author got the same bound in [14, Theorem 2.3] without the square-free requirement.

For $\mathcal{R}(\zeta_3;q) = \mathcal{R}(\zeta_3^2;q)$, we have $A(\frac{1}{3};n) = A(\frac{2}{3};n) = N(0,3;n) - N(1,3;n) = N(0,3;n) - N(2,3;n)$. With the notation $B_{\ell,u,c}(n,m)$ in (1.37), in [17, Proposition 5.1] Bringmann proved $A(\frac{1}{3};3n) < 0$, $A(\frac{1}{3};3n+1) > 0$ and $A(\frac{1}{3};3n+2) < 0$ for all $n \notin \{1,3,7\}$ based on the asymptotic formula [17, Theorem 1.1]:

$$A\left(\frac{1}{3};n\right) = \frac{4\sqrt{3}i}{(24n-1)^{\frac{1}{2}}} \sum_{3|k\leq\sqrt{n}} \frac{B_{1,3,k}(-n,0)}{\sqrt{k}} \sinh\left(\frac{\pi\sqrt{24k-1}}{6k}\right) + O_\varepsilon(n^\varepsilon).$$

Bringmann and Ono claimed in [21] that this formula, when summing up to infinity, is the exact formula for $A(\frac{1}{3};n)$.

This claim, which is exactly Theorem 1.11, has been proved by the author in the previous two sections. If we denote $R_3(n,x)$ as the tail sum as (1.36), then Bringmann's result gives $R_3(n,\sqrt{n}) \ll_\varepsilon n^\varepsilon$, while the author's Theorem 1.12 improves the bound as $R_3(n,\alpha\sqrt{n}) \ll_{\alpha,\varepsilon} n^{-\frac{1}{147}+\varepsilon}$.

# Chapter 7

# Partitions of rank modulo a prime $p \geq 5$

In this chapter we prove Theorem 1.14. We also prove (1.44) and (1.45) in the Remark to show that this formula matches the result by Bringmann [17] in (1.41). Further more, we obtain an extra coincidence and put it in the last section for future interests. Recall our notations in Section 2.4.

## 7.1 Vector-valued Maass-Poincaré series

We construct harmonic Maass forms via the so-called Maass-Poincaré series. For $s \in \mathbb{C}$, $k \in \mathbb{Z} + \frac{1}{2}$, and $z = x + iy \in \mathbb{C}$ with $x, y \in \mathbb{R}$, define

$$\mathcal{M}_s(y) := |y|^{-\frac{k}{2}} M_{\frac{k}{2}\operatorname{sgn} y,\ s-\frac{1}{2}}(|y|) \quad \text{and} \quad \varphi_{s,k}(z) := \mathcal{M}_s(4\pi y) e(x)$$

where $M_{\alpha,\beta}$ is the standard $M$-Whittaker function. One can check that $\varphi_{s,k}(z)$ is an eigenfunction of $\widetilde{\Delta}_k$ with eigenvalue $s(1-s) + \frac{k^2-2k}{4}$.

From now on we fix the prime $p \geq 5$ and focus on our $(p-1)$-dimensional weight $k = \frac{1}{2}$ multiplier system $\mu_p$ in Definition 2.11. For an integer $m \leq 0$, recall $m_{+\infty} = m - \frac{1}{24}$ defined in (2.35). Since we do not need the weight $-\frac{1}{2}$ case of $\overline{\mu_p}$ and only have $\mu_p$ in this section, we simply write $m_\infty$ instead of $m_{+\infty}$. For $m_\infty < 0$, we define the Maass-Poincaré series at the cusp $\infty$ by

$$\mathbf{P}_\infty(z; p, s, \tfrac{1}{2}, m, \mu_p) := \frac{2}{\sqrt{\pi}} \sum_{\ell=1}^{p-1} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(p)} \mu_p(\gamma)^{-1} \frac{\varphi_{s,k}(m_\infty \gamma z)}{(cz+d)^{\frac{1}{2}} \sin(\frac{\pi \ell}{p})} \mathfrak{e}_\ell. \tag{7.1}$$

By [21, Lemma 3.1], this series is absolutely and uniformly convergent on any compact subset of $\operatorname{Re} s > 1$. The transformation formula for $\mathbf{P}_\infty(z; p, s, \frac{1}{2}, m, \mu_p)$:

$$\mathbf{P}_\infty(\gamma_1 z; p, s, \tfrac{1}{2}, m, \mu_p) = \mu_p(\gamma_1)(Cz+D)^{\frac{1}{2}} \mathbf{P}_\infty(z; p, s, \tfrac{1}{2}, m, \mu_p) \quad \text{for } \gamma_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(p)$$

can be proved similarly as (3.17).

For an integer $r \geq 0$, recall the definition of $x_r$ in (2.37), the stabilizer group $\Gamma_0$ of the cusp 0 of $\Gamma_0(p)$: $\Gamma_0 = \{\pm\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right) : c \in p\mathbb{Z}\}$ and the scaling matrix $\sigma_0 = \left(\begin{smallmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{smallmatrix}\right)$. Recall the notations $\mathbf{X}_r$ and $X_{r,+0}^{(\ell)}$ in

(2.40). We denote $X_{r,0}^{(\ell)}$ instead of $X_{r,+0}^{(\ell)}$ for simplicity. Recall the notations $\triangleright\, r \,\triangleleft$ and $\triangleright\, a, r \,\triangleleft$ in (2.38).

For every integer $r \geq 0$, we define the Maass-Poincaré series at the cusp 0 by

$$\begin{aligned}&\mathbf{P}_0(z; p, s, \tfrac{1}{2}, r, \mu_p)\\ &:= \frac{2e(-\frac{1}{8})p^{\frac{1}{4}}}{\sqrt{\pi}} \sum_{\ell \in \triangleright r \triangleleft} \sum_{\substack{\gamma \in \Gamma_0 \backslash \Gamma_0(p) \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} \mu_p(\gamma)^{-1} \overline{\omega_{\frac{1}{2}}(\sigma_0^{-1}, \gamma)} \frac{\varphi_{s,k}\left(X_{r,0}^{(\ell)} \sigma_0^{-1} \gamma z\right)}{(-a\sqrt{p}z - b\sqrt{p})^{\frac{1}{2}}} \mathfrak{e}_\ell.\end{aligned} \tag{7.2}$$

Note that $\sigma_0^{-1}\gamma = \left(\begin{smallmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right)$. By [21, Lemma 3.1], the above series is absolutely and uniformly convergent on any compact subset of $\operatorname{Re} s > 1$. The transformation formula for $\mathbf{P}_0(z; p, s, \frac{1}{2}, r, \mu_p)$:

$$\mathbf{P}_0(\gamma_1 z; p, s, \tfrac{1}{2}, r, \mu_p) = \mu_p(\gamma_1)(Cz + D)^{\frac{1}{2}} \mathbf{P}_0(z; p, s, \tfrac{1}{2}, r, \mu_p) \quad \text{for } \gamma_1 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_0(p)$$

can be proved similarly as (3.31).

For convenience, we define the principal part of our vector-valued Maass-Poincaré series here. For a vector-valued smooth function $\mathbf{P}(z)$ which satisfies $\mathbf{P}(\gamma z) = \mu_p(\gamma)(cz+d)^{\frac{1}{2}}\mathbf{P}(z)$ for $\gamma \in \Gamma_0(p)$, if there exist $\mathbf{R}_\infty(z)$ and $\mathbf{R}_0(z)$ such that $R_\infty^{(\ell)}(z), R_0^{(\ell)}(z) \in \mathbb{C}[q^{-1}]$ for $1 \leq \ell \leq p-1$ and

$$P^{(\ell)}(z) - R_\infty^{(\ell)}(z) = O(e^{-Cy}), \quad (\sqrt{p}z)^{-\frac{1}{2}} P^{(\ell)}(-\tfrac{1}{pz}) - R_0^{(\ell)}(z) = O(e^{-Cy}) \quad \text{for } y \to \infty \text{ and some } C > 0,$$

then we call $\mathbf{R}_\infty(z)$ and $\mathbf{R}_0(z)$ the principal parts of $\mathbf{P}(z)$ at the cusps $\infty$ and 0 of $\Gamma_0(p)$, respectively. Moreover, if the Fourier expansion of $\mathbf{P}(z)$ can be written as

$$\mathbf{P}(z) = \sum_{\ell=1}^{p-1} \sum_{n \geq \mathfrak{M}} a_+^{(\ell)}(n) q^{n_\infty} \mathfrak{e}_\ell + \sum_{\ell=1}^{p-1} \sum_{n<0} a_-^{(\ell)}(n) \Gamma(\tfrac{1}{2}, 4\pi|n_\infty|y) q^{n_\infty} \mathfrak{e}_\ell,$$

for some $\mathfrak{M} \in \mathbb{Z}$, then the principal part of $\mathbf{P}(z)$ at the cusp $\infty$ is

$$\mathbf{R}_\infty(z) = \sum_{\ell=1}^{p-1} \sum_{\mathfrak{M} \leq n \leq 0} a_+^{(\ell)}(n) q^{n_\infty} \mathfrak{e}_\ell. \tag{7.3}$$

We take $n \leq 0$ because of $\alpha_\infty = \frac{1}{24} > 0$. The principal part of $\mathbf{P}(z)$ at the cusp 0 is clearly the principal part of $(\sqrt{p}z)^{-\frac{1}{2}}\mathbf{P}(-\frac{1}{pz})$ at the cusp $\infty$.

### 7.1.1 Fourier expansions of $\mathbf{P}_\infty$ at $\infty$

In this subsection, we compute the Fourier expansions of $\mathbf{P}_\infty(z; p, s, \frac{1}{2}, m, \mu_p)$ at $s = \frac{3}{4}$. It is important to note that we only have the absolute and uniform convergence for $\operatorname{Re} s > 1$ by definition. However, the Fourier expansion in the following theorem is guaranteed to be convergent by Proposition 3.14 when $s = \frac{3}{4}$. By analytic continuation, $\mathbf{P}_\infty(z; p, s, \frac{1}{2}, m, \mu_p)$ is convergent at $s = \frac{3}{4}$ and has the Fourier expansion as below. The proof of Proposition 3.14 is independent from this chapter.

There are similar arguments in [7, Proof of Theorem 3.1] for $p = 2$ and [14, Theorem 4.3] for $p = 3$.

**Proposition 7.1.** *When $m_\infty < 0$, the Maass-Poincaré series $\mathbf{P}_\infty(z; p, s, \frac{1}{2}, m, \mu_p)$ is convergent at $s = \frac{3}{4}$,* *and we have the following Fourier expansion:*

$$\mathbf{P}_\infty(z;p,\tfrac34,\tfrac12,m,\mu_p)=\sum_{\ell=1}^{p-1}\left(1-\frac{\Gamma(\frac12,4\pi|m_\infty|y)}{\sqrt{\pi}}\right)\frac{q^{m_\infty}}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell$$
$$+\sum_{n_\infty>0}\mathbf{B}_\infty(n)q^{n_\infty}+\sum_{n_\infty<0}\mathbf{B}'_\infty(n)\frac{\Gamma\left(\frac12,4\pi|n_\infty|y\right)}{\sqrt{\pi}}q^{n_\infty},$$

*where*

$$\left.\begin{matrix}\mathbf{B}_\infty(n)\\ \mathbf{B}'_\infty(n)\end{matrix}\right\}=2\pi e(-\tfrac18)\left|\frac{m_\infty}{n_\infty}\right|^{\frac14}\sum_{\ell=1}^{p-1}\sum_{N|c>0}\frac{\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}\left\{\begin{matrix}I_{\frac12}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right)\\ J_{\frac12}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right).\end{matrix}\right. \tag{7.4}$$

*Here* $\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)$ *is defined by* (2.44) *and its scalar value can be written as*

$$S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)=e(-\tfrac18)\sum_{\substack{d\ (\mathrm{mod}\ c)^*\\ ad\equiv1\ (\mathrm{mod}\ c)}}\frac{\overline{\mu(c,d,[a\ell],p)}}{\sin(\frac{\pi[a\ell]}{p})}e^{-\pi is(d,c)}e\left(\frac{ma+nd}{c}\right). \tag{7.5}$$

*Proof.* The following process is well-known and we provide details for readers to check. Recall the properties of Whittaker functions from (2.18) to (2.22).

The contribution to $\mathbf{P}_\infty(z;p,s,\frac12,m,\mu_p)$ from $c=0$ equals

$$\frac{2}{\sqrt{\pi}}\sum_{\ell=1}^{p-1}\csc(\tfrac{\pi\ell}{p})\varphi_{s,\frac12}(m_\infty z)\mathfrak{e}_\ell.$$

When $s=\frac34$, by (2.20), such contribution is

$$\frac{2}{\sqrt{\pi}}\sum_{\ell=1}^{p-1}\csc(\tfrac{\pi\ell}{p})\varphi_{\frac34,\frac12}(m_\infty z)\mathfrak{e}_\ell=\left(1-\frac{\Gamma(\frac12,4\pi|m_\infty|y)}{\sqrt{\pi}}\right)\sum_{\ell=1}^{p-1}\csc(\tfrac{\pi\ell}{p})e^{2\pi m_\infty z}\mathfrak{e}_\ell.$$

Recall (Definition 2.11) that $\mu_p(\left(\begin{smallmatrix}a&*\\ *&d\end{smallmatrix}\right))^{-1}$ maps the value at the $[a\ell]$-th entry to the $\ell$-th entry. Using the properties (1.9) for $\nu_\eta$ and Proposition 2.12 for $\mu_p$ and $M_p$, for $\mathrm{Re}\,s>0$, the contribution to $\mathbf{P}_\infty(z;p,s,\frac12,m,\mu_p)$ from some $c>0$ equals

$$\begin{aligned}&\frac{2}{\sqrt{\pi}}\sum_{\ell=1}^{p-1}\sum_{t\in\mathbb{Z}}\sum_{d(c)^*}\mu_p^{-1}\left(\begin{smallmatrix}a&b+ta\\ c&d+tc\end{smallmatrix}\right)(cz+d+tc)^{-\frac12}\csc(\tfrac{\pi\ell}{p})\\ &\quad\cdot\mathcal{M}_s\left(\frac{4\pi\tilde{m}y}{|cz+d+tc|^2}\right)e\left(\frac{m_\infty a}{c}-\mathrm{Re}\left(\frac{m_\infty}{c(cz+d+tc)}\right)\right)\mathfrak{e}_\ell\\ &=\frac{2}{\sqrt{\pi}c}\sum_{\ell=1}^{p-1}\sum_{d(c)^*}\frac{\overline{\mu(c,d,[a\ell],p)}}{\sin(\frac{\pi[a\ell]}{p})}\nu_\eta(\left(\begin{smallmatrix}a&b\\ c&d\end{smallmatrix}\right))e\left(\frac{m_\infty a}{c}\right)\sum_{t\in\mathbb{Z}}e\left(t\alpha_\infty\right)\left(z+\frac{d}{c}+t\right)^{-\frac12}\\ &\quad\cdot\mathcal{M}_s\left(\frac{4\pi\tilde{m}y}{c^2|z+\frac{d}{c}+t|^2}\right)e\left(-\frac{m_\infty}{c^2}\mathrm{Re}\left(\frac{1}{z+\frac{d}{c}+t}\right)\right)\mathfrak{e}_\ell.\end{aligned} \tag{7.6}$$

Here we use $\sum_{d(c)^*}$ to abbreviate the following summation condition: for $d \pmod{c}^*$, we choose $a$ by $ad \equiv 1 \pmod{c}$ and $b$ by $ad - bc = 1$.

Let

$$f(z) := \sum_{t\in\mathbb{Z}} \frac{e(t\alpha_\infty)}{(z+t)^{\frac12}} \mathcal{M}_s\left(\frac{4\pi m_\infty y}{c^2|z+t|^2}\right) e\left(-\frac{m_\infty}{c^2}\operatorname{Re}\left(\frac{1}{z+t}\right)\right).$$

Then $f(z)e(\alpha_\infty x)$ has period 1 and $f$ has Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}} a_y(n)e(n_\infty x) \quad \text{and} \quad f\left(z+\frac{d}{c}\right) = e\left(\frac{n_\infty d}{c}\right) f(z). \tag{7.7}$$

Here by [58, Proof of Theorem 1.9], we can compute

$$a_y(n) = \frac{e(-\frac18)\Gamma(2s)}{|4\pi m_\infty y|^{\frac14}\sqrt{c}} \cdot \begin{cases} \dfrac{2\pi}{\Gamma(s-\frac14)}\left|\dfrac{m_\infty}{n_\infty}\right|^{\frac12} W_{-\frac14,s-\frac12}(4\pi|n_\infty|y)J_{2s-1}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty<0; \\ \dfrac{2\pi}{\Gamma(s+\frac14)}\left|\dfrac{m_\infty}{n_\infty}\right|^{\frac12} W_{\frac14,s-\frac12}(4\pi n_\infty y)I_{2s-1}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty>0. \end{cases}$$

Thus, for $\operatorname{Re} s > 1$, we have the Fourier expansion of $\mathbf{P}_\infty(z;p,s,\frac12,m,\mu_p)$:

$$\mathbf{P}_\infty(z;p,s,\tfrac12,m,\mu_p) = \frac{2}{\sqrt{\pi}}\sum_{\ell=1}^{p-1}\frac{\varphi_{s,\frac12}(z)}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell + \sum_{n\in\mathbb{Z}} e^{2\pi i n_\infty z}\frac{2\Gamma(2s)e(-\frac18)|m_\infty|^{\frac14}}{\sqrt{\pi}|n_\infty|^{\frac12}|4\pi y|^{\frac14}}$$
$$\cdot\begin{cases} \dfrac{2\pi W_{-\frac14,s-\frac12}(4\pi|n_\infty|y)}{\Gamma(s-\frac14)}\displaystyle\sum_{\ell=1}^{p-1}\sum_{p|c>0}\dfrac{\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}J_{2s-1}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty<0; \\ \dfrac{2\pi W_{\frac14,s-\frac12}(4\pi n_\infty y)}{\Gamma(s+\frac14)}\displaystyle\sum_{\ell=1}^{p-1}\sum_{p|c>0}\dfrac{\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}I_{2s-1}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty>0. \end{cases}$$

For the right side of the expansion above, if we let $s=\frac34$, by (2.21) we get

$$\sum_{\ell=1}^{p-1}\left(1-\frac{\Gamma(\frac12,4\pi|m_\infty|y)}{\sqrt{\pi}}\right)\frac{e^{2\pi m_\infty z}}{\sin(\frac{\pi\ell}{p})}\mathfrak{e}_\ell + \sum_{n\in\mathbb{Z}} e^{2\pi i n_\infty z}\cdot 2\pi e(-\tfrac18)\left|\frac{m_\infty}{n_\infty}\right|^{\frac14}$$
$$\cdot\begin{cases} \dfrac{\Gamma(\frac12,4\pi|n_\infty|y)}{\sqrt{\pi}}\displaystyle\sum_{\ell=1}^{p-1}\sum_{p|c>0}\dfrac{\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}J_{\frac12}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty<0; \\ \displaystyle\sum_{\ell=1}^{p-1}\sum_{p|c>0}\dfrac{\mathbf{S}^{(\ell)}_{\infty\infty}(m,n,c,\mu_p)}{c}I_{\frac12}\left(\dfrac{4\pi|m_\infty n_\infty|^{\frac12}}{c}\right), & n_\infty>0. \end{cases}$$

By Proposition 3.14, the above expression is convergent. Therefore, by analytic continuation, the series $\mathbf{P}_\infty(z;p,s,\frac12,m,\mu_p)$ is convergent at $s=\frac34$ and has the Fourier expansion as above.

The last expression (7.5) is easily deduced by combining (2.44), Definition 2.11, and (1.18). □

### 7.1.2 Fourier expansion of $\mathbf{P}_0$ at $\infty$

In this subsection, we compute the Fourier expansions of $\mathbf{P}_0(z;p,s,\frac12,r,\mu_p)$ at $s=\frac34$. Also note that the convergence of the Fourier expansion in Proposition 7.2 is guaranteed by Proposition 3.14 when $s=\frac34$. Hence we have the convergence of $\mathbf{P}_0(z;p,s,\frac12,r,\mu_p)$ at $s=\frac34$ by analytic continuation. Recall our notations $\triangleright\, r \,\triangleleft$

and $\rhd a, r \lhd$ in (2.38) and $\alpha_{+0}^{(\ell)}$ in (2.36). Since we do not consider the weight $-\frac{1}{2}$ case of $\overline{\mu_p}$ here, we write $X_{r,0}^{(\ell)} = X_{r,+0}^{(\ell)}$ for simplicity.

**Proposition 7.2.** *For an integer $r \geq 0$, the series $\mathbf{P}_0(z; p, s, \frac{1}{2}, r, \mu_p)$ is convergent at $s = \frac{3}{4}$ and we have*

$$\mathbf{P}_0(z; p, \tfrac{3}{4}, \tfrac{1}{2}, r, \mu_p) = \sum_{n_\infty > 0} \mathbf{B}_0(n) q^{n_\infty} + \sum_{n_\infty < 0} \mathbf{B}_0'(n) \frac{\Gamma\left(\frac{1}{2}, 4\pi |n_\infty| y\right)}{\sqrt{\pi}} q^{n_\infty},$$

*where*

$$\left.\begin{matrix} \mathbf{B}_0(n) \\ \mathbf{B}_0'(n) \end{matrix}\right\} = 2\pi \sum_{\ell=1}^{p-1} \sum_{\substack{a>0:\, p\nmid a,\\ [a\ell] \in \rhd r \lhd}} \left| \frac{X_{r,0}^{([a\ell])}}{p n_\infty} \right|^{\frac{1}{4}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)}{a} \left\{ \begin{matrix} I_{\frac{1}{2}}\left( \frac{4\pi}{a} \left| \frac{X_{r,0}^{([a\ell])} n_\infty}{p} \right|^{\frac{1}{2}} \right) \\ J_{\frac{1}{2}}\left( \frac{4\pi}{a} \left| \frac{X_{r,0}^{([a\ell])} n_\infty}{p} \right|^{\frac{1}{2}} \right) \end{matrix} \right. . \tag{7.8}$$

*Here $\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p) = S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)\mathfrak{e}_\ell$ is defined in (2.49). If $[a\ell] \in \rhd r \lhd$, we have*

$$S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p) = e(-\tfrac{1}{8}) \sum_{\substack{b:\ b \pmod{a}^* \\ p|c,\, 0<c<pa \\ s.t.\ ad-bc=1}} \overline{\mu(c, d, [a\ell], p) e^{\pi i s(d,c)}} e\left( \frac{m_{r,0}^{([a\ell])} \cdot \frac{c}{p} - n_\infty b}{-a} + \frac{a+d}{24c} \right); \tag{7.9}$$

*if $[a\ell] \notin \rhd r \lhd$, we have $S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p) = 0$.*

*Remark.* In the Fourier expansion, when $\ell$ is fixed, for the summation on $a$ we only select $a$ such that $p \nmid a$ and $[a\ell] \in \rhd r \lhd$. It is also important to note that the denominator in the last exponential term in (7.9) is $-a$, which is negative.

*Proof.* Recall the double coset decomposition (3.32) and the choice of $\gamma_2$ below it:

$$\sigma_0^{-1} \Gamma_0(p) \sigma_\infty = \bigcup_{\substack{a>0\\ p\nmid a}} \bigcup_{b \pmod{a}^*} \Gamma_\infty \begin{pmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{pmatrix} \Gamma_\infty,$$

$$\sigma_0^{-1}(\Gamma_0 \setminus \Gamma_0(p)) = \{\sigma_0^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) :\ a > 0,\ p \nmid a,\ b \pmod{a}^*,\ t \in \mathbb{Z}\}.$$

One can check that for $c \geq 0$ and $a > 0$, we have

$$w_{\frac{1}{2}}(\sigma_0^{-1}, \sigma_0 \gamma)(-a\sqrt{p} z - b\sqrt{p})^{\frac{1}{2}} = \left( \frac{-a\sqrt{p}\, z - b\sqrt{p}}{cz+d} \right)^{\frac{1}{2}} (cz+d)^{\frac{1}{2}} = -ip^{\frac{1}{4}} (az+b)^{\frac{1}{2}}.$$

In the double coset decomposition, we can take the representative $\left(\begin{smallmatrix} c/\sqrt{p} & * \\ -a\sqrt{p} & * \end{smallmatrix}\right)$ with $a > 0$ and $c \geq 0$ because

$\left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} c/\sqrt{p} & * \\ -a\sqrt{p} & * \end{smallmatrix}\right) = \left(\begin{smallmatrix} (c-\beta ap)/\sqrt{p} & * \\ -a\sqrt{p} & * \end{smallmatrix}\right)$ for any $\beta \in \mathbb{Z}$. Then from (7.2), for $\operatorname{Re} s > 1$ we have

$$\begin{aligned}
&\mathbf{P}_0(z; p, s, \tfrac{1}{2}, r, \mu_p) \\
&= \frac{2e(-\frac{1}{8})p^{\frac{1}{4}}}{\sqrt{\pi}} \sum_{\ell \in \triangleright r \triangleleft} \sum_{\substack{\gamma = \left(\begin{smallmatrix} \frac{c}{\sqrt{p}} & \frac{d}{\sqrt{p}} \\ -a\sqrt{p} & -b\sqrt{p} \end{smallmatrix}\right) \\ \gamma \in \Gamma_\infty \backslash \sigma_0^{-1}\Gamma_0(p)}} \mu_p(\sigma_0\gamma)^{-1} \overline{w_{\frac{1}{2}}(\sigma_0^{-1}, \sigma_0\gamma)} \frac{\varphi_{s,\frac{1}{2}}(X_{r,0}^{(\ell)}\gamma z)}{(-a\sqrt{p}z - b\sqrt{p})^{\frac{1}{2}}} \mathfrak{e}_\ell \\
&= \frac{2e(\frac{1}{8})}{\sqrt{\pi}} \sum_{\ell \in \triangleright r \triangleleft} \sum_{\substack{a > 0 \\ p \nmid a}} \sum_{b(a)^*} \sum_{t \in \mathbb{Z}} \mu_p\left(\left(\begin{smallmatrix} a & b+ta \\ c & d+tc \end{smallmatrix}\right)\right)^{-1} \frac{\varphi_{s,\frac{1}{2}}(X_{r,0}^{(\ell)}\gamma z)}{(az+b+ta)^{\frac{1}{2}}} \mathfrak{e}_\ell.
\end{aligned}$$

Here and below we use $\sum_{b(a)^*}$ to abbreviate the following summation condition: $c$ and $d$ are determined by $p|c$, $0 < c < pa$, and $ad - bc = 1$.

Observe that $\gamma z = \frac{cz+d}{-paz-pb} = -\frac{c}{pa} - \frac{1}{pa(az+b)}$, $\mu(c, d+tc, \ell, p) = \mu(c, d, \ell, p)$ for all $\ell$ and $t$, and $\nu_\eta\left(\left(\begin{smallmatrix} a & b+ta \\ c & d+tc \end{smallmatrix}\right)\right) = \nu_\eta\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) e(t\alpha_\infty)$ by (1.9). The contribution from a single $a$ for $p \nmid a$ is then

$$\begin{aligned}
&\frac{2e(\frac{1}{8})}{\sqrt{\pi a}} \sum_{\ell \in \triangleright r \triangleleft} \sum_{b(a)^*} \overline{\mu(c, d, \ell, p)} \nu_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) e\left(\frac{-X_{r,0}^{(\ell)}c}{pa}\right) \sum_{t \in \mathbb{Z}} e(t\alpha_\infty) \\
&\quad \cdot \left(z + \tfrac{b}{a} + t\right)^{-\frac{1}{2}} \mathcal{M}_s\left(\frac{4\pi X_{r,0}^{(\ell)} y}{pa^2|z + \frac{b}{a} + t|^2}\right) e\left(\frac{-X_{r,0}^{(\ell)}}{pa^2} \operatorname{Re}\left(\frac{1}{z + \frac{b}{a} + t}\right)\right) \mathfrak{e}_{[d\ell]} \\
&= \frac{2e(\frac{1}{8})}{\sqrt{\pi a}} \sum_{\ell \in \triangleright a, r \triangleleft} \sum_{b(a)^*} \overline{\mu(c, d, [a\ell], p)} \nu_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) e\left(\frac{-X_{r,0}^{([a\ell])}c}{pa}\right) \sum_{t \in \mathbb{Z}} e(t\alpha_\infty) \\
&\quad \cdot \left(z + \tfrac{b}{a} + t\right)^{-\frac{1}{2}} \mathcal{M}_s\left(\frac{4\pi X_{r,0}^{([a\ell])} y}{pa^2|z + \frac{b}{a} + t|^2}\right) e\left(\frac{-X_{r,0}^{([a\ell])}}{pa^2} \operatorname{Re}\left(\frac{1}{z + \frac{b}{a} + t}\right)\right) \mathfrak{e}_\ell.
\end{aligned}$$

Here we have changed $[d\ell]$ to $\ell$, hence $\ell$ to $[a\ell]$ and $\ell \in \triangleright r \triangleleft$ to $\ell \in \triangleright a, r \triangleleft$.

As in the case of $\mathbf{P}_\infty$ in Proposition 7.1, we let

$$f(z) := \sum_{t \in \mathbb{Z}} \frac{e(t\alpha_\infty)}{(z+t)^k} \mathcal{M}_s\left(\frac{4\pi X_{r,0}^{([a\ell])} y}{pa^2|z+t|^2}\right) e\left(\frac{-X_{r,0}^{([a\ell])}}{pa^2} \operatorname{Re}\left(\frac{1}{z+t}\right)\right).$$

Then $f(z)e(\alpha_\infty x)$ has period 1 and $f$ has Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} a_y(n) e(n_\infty x) \quad \text{and} \quad f\left(z + \frac{b}{a}\right) = e\left(\frac{n_\infty b}{a}\right) f(z). \tag{7.10}$$

Here by [58, Proof of Theorem 1.9], we have

$$a_y(n) = \frac{e(-\frac{1}{8})\Gamma(2s)}{\left|4\pi X_{r,0}^{([a\ell])} y\right|^{\frac{1}{4}} p^{\frac{1}{4}} \sqrt{a}} \cdot \begin{cases} \dfrac{2\pi}{\Gamma(s-\frac{1}{4})} \left|\dfrac{X_{r,0}^{([a\ell])}}{n_\infty}\right|^{\frac{1}{2}} W_{-\frac{1}{4},s-\frac{1}{2}}(4\pi|n_\infty|y) J_{2s-1}\left(\dfrac{4\pi}{a}\left|\dfrac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty < 0; \\ \dfrac{2\pi}{\Gamma(s+\frac{1}{4})} \left|\dfrac{X_{r,0}^{([a\ell])}}{n_\infty}\right|^{\frac{1}{2}} W_{\frac{1}{4},s-\frac{1}{2}}(4\pi n_\infty y) I_{2s-1}\left(\dfrac{4\pi}{a}\left|\dfrac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty > 0. \end{cases}$$

Thus, the Fourier expansion of $\mathbf{P}_0(z; p, s, \frac{1}{2}, r, \mu_p)$ at the cusp $\infty$ for $\operatorname{Re} s > 1$ is:

$$\mathbf{P}_0(z; p, s, \tfrac{1}{2}, \mathbf{X}_r, \mu_p) = \sum_{\ell=1}^{p-1} \sum_{n\in\mathbb{Z}} e^{2\pi i n_\infty z} \cdot \frac{2\Gamma(2s)}{\sqrt{\pi}|4\pi y|^{\frac{1}{4}} p^{\frac{1}{4}} |n_\infty|^{\frac{1}{2}}}$$
$$\cdot \begin{cases} \displaystyle\sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]\in \triangleright r \triangleleft}} \frac{2\pi W_{-\frac{1}{4},s-\frac{1}{2}}(4\pi|n_\infty|y)}{\Gamma(s-\frac{1}{4})\left|X_{r,0}^{([a\ell])}\right|^{-\frac{1}{4}}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)}{a} J_{\frac{1}{2}}\left(\frac{4\pi}{a}\left|\frac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty < 0; \\ \displaystyle\sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]\in \triangleright r \triangleleft}} \frac{2\pi W_{\frac{1}{4},s-\frac{1}{2}}(4\pi n_\infty y)}{\Gamma(s+\frac{1}{4})\left|X_{r,0}^{([a\ell])}\right|^{-\frac{1}{4}}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)}{a} I_{\frac{1}{2}}\left(\frac{4\pi}{a}\left|\frac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty > 0. \end{cases}$$

For the right side of the expansion above, if we let $s = \frac{3}{4}$, by (2.21) we get

$$\sum_{\ell=1}^{p-1} \sum_{n\in\mathbb{Z}} 2\pi e^{2\pi i n_\infty z}$$
$$\cdot \begin{cases} \displaystyle\sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]\in \triangleright r \triangleleft}} \frac{\Gamma(\frac{1}{2}, 4\pi|n_\infty|y)}{\sqrt{\pi}} \left|\frac{X_{r,0}^{([a\ell])}}{p n_\infty}\right|^{\frac{1}{4}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)}{a} J_{\frac{1}{2}}\left(\frac{4\pi}{a}\left|\frac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty < 0; \\ \displaystyle\sum_{\substack{a>0:\, p\nmid a,\\ [a\ell]\in \triangleright r \triangleleft}} \left|\frac{X_{r,0}^{([a\ell])}}{p n_\infty}\right|^{\frac{1}{4}} \frac{\mathbf{S}_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p)}{a} I_{\frac{1}{2}}\left(\frac{4\pi}{a}\left|\frac{X_{r,0}^{([a\ell])}}{p}\cdot n_\infty\right|^{\frac{1}{2}}\right), & n_\infty > 0. \end{cases}$$

By Proposition 3.14, where we take $\mathbf{m} = \mathbf{X}_r \le 0$, the above expression is convergent. Therefore, by analytic continuation, the series $\mathbf{P}_0(z; p, s, \frac{1}{2}, r, \mu_p)$ is convergent at $s = \frac{3}{4}$ and has the Fourier expansion as above. The expression (7.9) is deduced by combining (2.48), (2.45), Definition 2.11, and (1.18). □

We combine the properties of the Maass-Poincaré series in the following proposition.

**Proposition 7.3.** *Let* $\mathbf{P}(z)$ *denote either of* $\mathbf{P}_\infty(z) := \mathbf{P}_\infty(z; p, \frac{3}{4}, \frac{1}{2}, m, \mu_p)$ *or* $\mathbf{P}_0(z) := \mathbf{P}_0(z; p, \frac{3}{4}, \frac{1}{2}, r, \mu_p)$. *Then,*

*(1) For all* $\gamma \in \Gamma_0(p)$, $\mathbf{P}(\gamma z) = \mu_p(\gamma)(cz+d)^{\frac{1}{2}}\mathbf{P}(z)$.

*(2) For* $1 \le \ell \le p-1$, *the* $\ell$*-th entry* $P^{(\ell)}(z)$ *of* $\mathbf{P}(z)$ *is a harmonic Maass form in* $H_{\frac{1}{2}}(\Gamma_0(p^2)\cap\Gamma_1(p), \overline{\nu_\eta})$.

*(3) For* $1 \le \ell \le p-1$, $P^{(\ell)}(24z)$ *is a harmonic Maass form in* $H_{\frac{1}{2}}(\Gamma_1(576p^2), \nu_\theta)$.

*(4) The principal part of* $\mathbf{P}_\infty(z;p,\frac{3}{4},\frac{1}{2},m,\mu_p)$ *at the cusp* $\infty$ *of* $\Gamma_0(p)$ *is*

$$\sum_{\ell=1}^{p-1} q^{m_\infty}\csc(\tfrac{\pi\ell}{p})\mathfrak{e}_\ell,$$

*and at the cusp* $0$ *of* $\Gamma_0(p)$ *is* $\mathbf{0}$.

*(5) For every integer* $r\geq 0$*, the principal part of* $\mathbf{P}_0^{(\ell)}(z;p,\frac{3}{4},\frac{1}{2},r,\mu_p)$ *at the cusp* $\infty$ *of* $\Gamma_0(p)$ *is* $\mathbf{0}$*, and at the cusp* $0$ *of* $\Gamma_0(p)$ *is*

$$e(-\tfrac{1}{8})p^{\frac{1}{4}}\sum_{\ell\in\rhd r\lhd} q^{X_{r,0}^{(\ell)}}\mathfrak{e}_\ell.$$

*Proof.* First we prove (1) and (2). We have discussed the transformation laws of $\mathbf{P}_\infty(z;p,s,\frac{1}{2},m,\mu_p)$ and $\mathbf{P}_0(z;p,s,\frac{1}{2},r,\mu_p)$ directly after their definitions. Since we have proved their convergence at $s=\frac{3}{4}$, by analytic continuation, the transformation laws are kept. When we focus on each entry $P^{(\ell)}(z)$ and $G^{(\ell)}(z):=P^{(\ell)}(24z)$, the transformation laws

$$P^{(\ell)}(\gamma z)=\overline{\nu_\eta}(\gamma)(cz+d)^{\frac{1}{2}}P^{(\ell)}(z),\quad \gamma\in\Gamma_0(p^2)\cap\Gamma_1(p),$$
$$G^{(\ell)}(\gamma z)=\nu_\theta(\gamma)(cz+d)^{\frac{1}{2}}G^{(\ell)}(z),\quad \gamma\in\Gamma_1(576p^2)$$

follow from Lemma 2.15.

Recall the definition for the principal parts before (7.3). For (3), since $\varphi_{s,k}(z)$ is an eigenfunction of $\widetilde{\Delta}_k$ with eigenvalue $s(1-s)+\frac{k^2-2k}{4}$, when $k=\frac{1}{2}$ and $s=\frac{3}{4}$, we have $\widetilde{\Delta}_{\frac{1}{2}}\varphi_{\frac{3}{4},\frac{1}{2}}(z)=0$. Therefore, we have $\widetilde{\Delta}_{\frac{1}{2}}\mathbf{P}(z)=0$.

For (4), the principal part of $\mathbf{P}_\infty$ at the cusp $\infty$ can be read from Proposition 7.1. Note that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}\sigma_0=\begin{pmatrix} b\sqrt{p} & -a/\sqrt{p}\\ d\sqrt{p} & -c/\sqrt{p}\end{pmatrix}$$

and $d\neq 0$ for $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(p)$. As in Proposition 7.2, we can conclude that the principal part of $\mathbf{P}_\infty$ at the cusp 0 is $\mathbf{0}$.

For (5), the principal part of $\mathbf{P}_0$ at the cusp $\infty$ is just $\mathbf{0}$ from Proposition 7.2. To compute its principal part at 0, recall (7.2) for the definition and (6.7). The Fourier expansion of $\mathbf{P}_0$ at the cusp 0 is given by

$$(\sqrt{p}\,z)^{-\frac{1}{2}}\mathbf{P}_0(\sigma_0 z;p,\tfrac{3}{4},\tfrac{1}{2},r,\mu_p).$$

Then the contribution from $c=0$ equals

$$\begin{aligned}\frac{2e(-\frac{1}{8})p^{\frac{1}{4}}}{\sqrt{\pi}}(\sqrt{p}\,z)^{-\frac{1}{2}}\sum_{\ell\in\rhd r\lhd}\frac{\varphi_{\frac{3}{4},\frac{1}{2}}(X_{r,0}^{(\ell)}z)}{(\sqrt{p}\,z)^{-\frac{1}{2}}}\mathfrak{e}_\ell\\ =e(-\tfrac{1}{8})p^{\frac{1}{4}}\sum_{\ell\in\rhd r\lhd}\left(1-\frac{\Gamma(\frac{1}{2},4\pi|X_{r,0}^{(\ell)}|y)}{\sqrt{\pi}}\right)q^{X_{r,0}^{(\ell)}}\mathfrak{e}_\ell,\end{aligned}$$

and (5) follows. $\square$

## 7.2 Proof of Theorem 1.14

Fix a prime $p \geq 5$ and let $1 \leq \ell \leq p-1$. Recall the definition of $\mathcal{G}_1(\frac{\ell}{p}; z)$ in (2.30) and $\mathcal{G}_2(\frac{\ell}{p}; z)$ in (2.31). Also recall that the holomorphic part of $\mathcal{G}_1(\frac{\ell}{p}; z)$ has Fourier expansion

$$\csc\left(\frac{\pi\ell}{p}\right)\sum_{n=0}^{\infty} A\left(\frac{\ell}{p}; n\right) q^{n-\frac{1}{24}}. \tag{7.11}$$

Let the vector-valued function $\mathbf{G}_1(z;p)$ be defined as

$$\mathbf{G}_1(z;p) := \sum_{\ell=1}^{p-1} \mathcal{G}_1\left(\frac{\ell}{p}; z\right)\mathfrak{e}_\ell. \tag{7.12}$$

By Proposition 2.10, $\mathbf{G}_1(\cdot;p)$ has the property

$$\mathbf{G}_1(\gamma z;p) = \mu_p(\gamma)(cz+d)^{\frac{1}{2}}\mathbf{G}_1(z;p), \quad \text{for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p). \tag{7.13}$$

By (7.11) and (7.3), the principal part of $\mathbf{G}_1(z;p)$ at the cusp $\infty$ of $\Gamma_0(p)$ is

$$\csc\left(\frac{\pi\ell}{p}\right)\sum_{\ell=1}^{p-1} q^{-\frac{1}{24}}\mathfrak{e}_\ell. \tag{7.14}$$

By [18, (3.13)], the behavior of $\mathcal{G}_1$ at the cusp 0 is given by

$$(\sqrt{p}\,z)^{-\frac{1}{2}}\mathcal{G}_1\left(\frac{\ell}{p}; \sigma_0 z\right) = (\sqrt{p}\,z)^{-\frac{1}{2}}\mathcal{G}_1\left(\frac{\ell}{p}; -\frac{1}{pz}\right) = e(-\tfrac{1}{8})p^{\frac{1}{4}}\mathcal{G}_2\left(\frac{\ell}{p}; pz\right). \tag{7.15}$$

By the discussion after (7.3), the principal part of $\mathbf{G}_1(z;p)$ at the cusp 0 can be derived from the principal parts of $\mathcal{G}_2(\frac{\ell}{p}; pz)$ at the cusp $\infty$ for $1 \leq \ell \leq p-1$.

Recall that $\varepsilon_2$ is defined in (2.29) by

$$\varepsilon_2\left(\frac{\ell}{p}; z\right) = \begin{cases} 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2 + \frac{\ell}{2p} - \frac{1}{24}}, & \frac{\ell}{p} \in (0, \frac{1}{6}), \\ 2q^{-\frac{3}{2}\left(1-\frac{\ell}{p}\right)^2 + \frac{1}{2}\left(1-\frac{\ell}{p}\right) - \frac{1}{24}}, & \frac{\ell}{p} \in (\frac{5}{6}, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Here $\frac{1}{6}$ is the only root of the quadratic equation $-\frac{3}{2}x^2 + \frac{1}{2}x - \frac{1}{24} = 0$ hence the order of $\varepsilon_2(\frac{\ell}{p}; z)$ at $\infty$ is less than 0 in the first two cases. By (2.31), the holomorphic part of $\mathcal{G}_2(\frac{\ell}{p}; z)$ is

$$\varepsilon_2\left(\frac{\ell}{p}; z\right) + 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2 + \frac{3\ell}{2p} - \frac{1}{24}} M\left(\frac{\ell}{p}; z\right). \tag{7.16}$$

Recall (2.37) that $x_r$ is the only solution in $(0, \frac{1}{2})$ of the quadratic equation

$$-\frac{3}{2}x^2 + \left(\frac{1}{2} + r\right)x - \frac{1}{24} = 0.$$

Now $x_0 = \frac{1}{6}$ and the contribution from $\varepsilon_2(\frac{\ell}{p}; z)$ to the principal part of $\mathcal{G}_2(\frac{\ell}{p}; z)$ at $\infty$ is:

$$\begin{cases} 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2 + \frac{\ell}{2p} - \frac{1}{24}} & \text{when } 0 < \frac{\ell}{p} < x_0, \\ 2q^{-\frac{3}{2}\left(1-\frac{\ell}{p}\right)^2 + \frac{1}{2}\left(1-\frac{\ell}{p}\right) - \frac{1}{24}} & \text{when } 1 - x_0 < \frac{\ell}{p} < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7.17}$$

For the principal part of $\mathcal{G}_2(\frac{\ell}{p}; z)$ at $\infty$ contributed from the part other than $\varepsilon_2$, we need the Fourier expansion of $M(\frac{\ell}{p}; z)$ defined in (2.26):

**Lemma 7.4.** *Let $p \geq 5$ be a prime. When $1 \leq \ell \leq \frac{p-1}{2}$, the first few terms of the Fourier expansion of $M(\frac{\ell}{p}; z)$ are*

$$M\left(\frac{\ell}{p}; z\right) = \sum_{T=0}^{\lfloor \frac{p}{2\ell} \rfloor} q^{\frac{T\ell}{p}} + O(q^{\frac{1}{2}}).$$

*When $\frac{p+1}{2} \leq \ell \leq p-1$, we have $1 \leq p - \ell \leq \frac{p-1}{2}$ and the first few terms of the Fourier expansion of $M(\frac{\ell}{p}; z)$ are*

$$M\left(\frac{\ell}{p}; z\right) = \sum_{T=0}^{\lfloor \frac{p}{2(p-\ell)} \rfloor} q^{T\left(1-\frac{\ell}{p}\right)} + O(q^{\frac{1}{2}}).$$

*Proof.* It suffices to prove the first case $1 \leq \ell \leq \frac{p-1}{2}$ because $\mathcal{M}(\frac{\ell}{p}; z) = \mathcal{M}(1 - \frac{\ell}{p}; z)$ by (2.28). We have:

$$\begin{aligned} M\left(\frac{\ell}{p}; z\right) &= \prod_{j=1}^{\infty}(1-q^j)^{-1} \sum_{n\in\mathbb{Z}} \frac{(-1)^n q^{n+\frac{\ell}{p}}}{1-q^{n+\frac{\ell}{p}}} q^{\frac{3}{2}n^2 + \frac{3}{2}n} \\ &= \left(1 + q + 2q^2 + O(q^3)\right) \left( \sum_{n\geq 0} (-1)^n q^{n+\frac{\ell}{p}} q^{\frac{3}{2}n^2+\frac{3}{2}n} \sum_{T=0}^{\infty} q^{T(n+\frac{\ell}{p})} \right. \\ &\quad \left. + \sum_{\substack{n<0 \\ m=-n}} (-1)^{m+1} q^{\frac{3}{2}m^2 - \frac{3}{2}m} \sum_{T=1}^{\infty} q^{T(m-\frac{\ell}{p})} \right) \\ &= (1 + O(q)) \left( \left( q^{\frac{\ell}{p}} \sum_{T=0}^{\infty} q^{\frac{T\ell}{p}} + O(q) \right) + \left( 1 + \sum_{T=1}^{\infty} q^{T(1-\frac{\ell}{p})} + O(q) \right) \right) \\ &= \sum_{T=0}^{\lfloor \frac{p}{2\ell} \rfloor} q^{\frac{T\ell}{p}} + O(q^{\frac{1}{2}}). \end{aligned}$$

□

**Proposition 7.5.** *Let $p \geq 5$ be a prime, let $x_r$ be defined in (2.37), which is the only solution in $(0, \frac{1}{2})$ of the quadratic equation*

$$-\frac{3}{2}x^2 + \left(\frac{1}{2} + r\right)x - \frac{1}{24} = 0$$

*and let $R$ be the maximal integer such that $x_R^{-1} < p$. Then the sequence $\{x_r : r \geq 0\}$ is strictly decreasing and the principal part of $\mathcal{G}_2(\frac{\ell}{p}; z)$ contributed from the term involving $\mathcal{M}(\frac{\ell}{p}; z)$ (i.e. the part other than $\varepsilon_2(\frac{\ell}{p}; z)$)*

*equals*

$$\sum_{r=1}^{R}\begin{cases} 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2+\left(\frac{1}{2}+r\right)\frac{\ell}{p}-\frac{1}{24}} & \textit{when } 0<\frac{\ell}{p}<x_r, \\ 2q^{-\frac{3}{2}\left(1-\frac{\ell}{p}\right)^2+\left(\frac{1}{2}+r\right)\left(1-\frac{\ell}{p}\right)-\frac{1}{24}} & \textit{when } 1-x_r<\frac{\ell}{p}<1 \\ 0 & \textit{otherwise.} \end{cases} \tag{7.18}$$

*Proof.* By (7.16), we see that the term

$$q^{-\frac{3}{2}(\frac{\ell}{p})^2+\frac{\ell}{2p}-\frac{1}{24}}\cdot q^{\frac{r\ell}{p}}$$

contributes to the principal part if and only if $0<\frac{\ell}{p}<x_r$, (otherwise the exponent will be positive). The analogous case also holds for $1-\frac{\ell}{p}$ and $1-x_r$.

By Lemma 7.4, to ensure that the Fourier coefficient of $q^{\frac{r\ell}{p}}$ in the Fourier expansion of $M(\frac{\ell}{p};z)$ is 1, it suffices to show that $\frac{r\ell}{p}<\frac{1}{2}$ for $0<\frac{\ell}{p}<x_r$. Since $x_r\in(0,\frac{1}{2})$, we have

$$rx_r=\frac{3}{2}x_r^2-\frac{1}{2}x_r+\frac{1}{24}=\frac{3}{2}\left(x_r-\frac{1}{6}\right)^2<\frac{3}{2}\times\left(\frac{1}{2}-\frac{1}{6}\right)^2=\frac{1}{6}.$$

Thus $\frac{r\ell}{p}<rx_r<\frac{1}{6}<\frac{1}{2}$. The analogous case for $1-\frac{\ell}{p}$ can be proved in a similar way. ☐

Combining (7.17) and (7.18), we get the principal part of $\mathcal{G}_2(\frac{\ell}{p};z)$ at the cusp $\infty$:

$$\sum_{r=1}^{R}\begin{cases} 2q^{-\frac{3}{2}\left(\frac{\ell}{p}\right)^2+\left(\frac{1}{2}+r\right)\frac{\ell}{p}-\frac{1}{24}} & \text{when } 0<\frac{\ell}{p}<x_r, \\ 2q^{-\frac{3}{2}\left(1-\frac{\ell}{p}\right)^2+\left(\frac{1}{2}+r\right)\left(1-\frac{\ell}{p}\right)-\frac{1}{24}} & \text{when } 1-x_r<\frac{\ell}{p}<1 \\ 0 & \text{otherwise,} \end{cases} \tag{7.19}$$

where $R$ is the maximal integer such that $x_R^{-1}<p$.

*Remark.* Here we give a hint about the relation between $r$ and the prime $p$. Since $x_0=\frac{1}{6}$, when $p\le 5$, there is no principal part of $\mathcal{G}_2(\frac{\ell}{p};z)$ at the cusp $\infty$, hence no principal part of $\mathbf{G}_1$ at the cusp 0. Since $1/x_1=34.9706\cdots$, for $7\le p\le 31$, we only have $r=0$. Here is a table for first few conditions, where $[a,b]$ means the set of primes $p$ for $a\le p\le b$.

| | | | | | | |
|---|---|---|---|---|---|---|
| Range of $p$ | $p=5$ | $[7,31]$ | $[37,59]$ | $[61,83]$ | $[89,107]$ | $[109,131]$ |
| Allowed $r$ | No $r$ | $r=0$ | $r\le 1$ | $r\le 2$ | $r\le 3$ | $r\le 4$ |

Recall Proposition 7.3 about the principal parts of the Maass-Poincaré series. To match the principal part at the cusp $\infty$, we take $\mathbf{P}_\infty(z;p,\frac{3}{4},\frac{1}{2},0,\mu_p)$ due to (7.14).

For the cusp 0, we recall the definition of $\mathbf{X}_r$ in (2.40) and have

$$X_r^{(\ell)}:=\begin{cases} \left\lceil -\frac{3\ell^2}{2p}+(\frac{1}{2}+r)\ell-\frac{p}{24}\right\rceil, & \text{when } 0<\frac{\ell}{p}<x_r, \\ \left\lceil -\frac{3p}{2}(1-\frac{\ell}{p})^2+(\frac{1}{2}+r)p(1-\frac{\ell}{p})-\frac{p}{24}\right\rceil, & \text{when } 1-x_r<\frac{\ell}{p}<1, \\ 0, & \text{otherwise and will never be used,} \end{cases} \tag{7.20}$$

where $\lceil x\rceil$ is the smallest integer $\ge x$. Moreover, recalling $x_r$ and $\alpha_0^{(\ell)}$ (denoted as $\alpha_{+0}^{(\ell)}$ in (2.36) and (2.39)), we see that

$$X_{r,0}^{(\ell)}=-p\delta_{\ell,p,1,r},\quad X_r^{([a\ell])}=\lceil -p\delta_{\ell,p,a,r}\rceil,\quad X_{r,0}^{([a\ell])}=-p\delta_{\ell,p,a,r},$$

and $X_{r,0}^{(\ell)}$ match the order of the principal part of $\mathcal{G}_2(\frac{\ell}{p}; pz)$ in (7.19). Combining Proposition 7.3, (7.14), (7.15), and (7.19), we conclude the following proposition.

**Proposition 7.6.** *With the choice of $\mathbf{X}_r$ in (2.40), the principal parts of*

$$\mathbf{G}_1(z;p) - \mathbf{P}_\infty(z;p,\tfrac{3}{4},\tfrac{1}{2},0,\mu_p) - 2\sum_{\substack{r\geq 0\\ x_r^{-1}<p}} \mathbf{P}_0(z;p,\tfrac{3}{4},\tfrac{1}{2},r,\mu_p)$$

*are zero for both cusps $\infty$ and $0$ of $\Gamma_0(p)$.*

Now we start to prove Theorem 1.14.

**Lemma 7.7.** *For $\mathbf{X}_r$ defined in (2.40), the function*

$$\mathbf{G}(z) := \mathbf{G}_1(z;p) - \mathbf{P}_\infty(z;p,\tfrac{3}{4},\tfrac{1}{2},0,\mu_p) - 2\sum_{\substack{r\geq 0\\ x_r^{-1}<p}} \mathbf{P}_0(z;p,\tfrac{3}{4},\tfrac{1}{2},r,\mu_p)$$

*is a holomorphic modular form of weight $\frac{1}{2}$ on $(\Gamma_0(p),\mu_p)$, i.e. $\mathbf{G}(z)\in M_{\frac{1}{2}}(\Gamma_0(p),\mu_p)$.*

*Proof.* Recall Notation 2.5. By Lemma 2.15, (2.30), and Proposition 7.3, $G^{(\ell)}(z)$ is a harmonic Maass form in $H_{\frac{1}{2}}(\Gamma_0(p^2)\cap\Gamma_1(p),\overline{\nu_\eta})$ whose Fourier exponents are supported on $n-\frac{1}{24}$ for $n\in\mathbb{Z}$.

Since the principal part of $\mathbf{G}(z)$ is zero for both cusps $\infty$ and $0$ of $\Gamma_0(p)$, the principal part of $G^{(\ell)}(z)$ for every cusp of $\Gamma_0(p^2)\cap\Gamma_1(p)$ is zero. By Proposition 7.3, we know that

$$G^{(\ell)}(24z)\in H_{\frac{1}{2}}(\Gamma_1(576p^2),\nu_\theta)$$

with Fourier exponents supported on $24n-1$. We also have that the principal part of $G^{(\ell)}(24z)$ for every cusp of $\Gamma_1(p^2)$ is still zero. By [21, Lemma 2.3], $G^{(\ell)}(24z)$ is a holomorphic modular form in $M_{\frac{1}{2}}(\Gamma_1(576p^2),\overline{\nu_\theta})$.

Since $\mathbf{G}(z)$ follows the modular transformation law on $(\Gamma_0(p),\mu_p)$ and each entry $G^{(\ell)}(z)$ is holomorphic, we get the desired result. □

By Lemma 2.16, since $G^{(\ell)}(24z)$ has Fourier coefficients only supported on $24n-1$ for $n\geq 1$, combining the above lemma with Lemma 2.17 we have

**Corollary 7.8.** $\mathbf{G}(z)=\mathbf{0}$.

*Proof of Theorem 1.14.* The theorem follows directly by combining Corollary 7.8, Proposition 7.1 and Proposition 7.2. Note that the $n$-th Fourier coefficient of $G^{(\ell)}(z)$ is $\csc(\frac{\pi\ell}{p})A(\frac{\ell}{p};n)$, hence we need to multiply the Fourier expansion of the Maass-Poincaré series by $\sin(\frac{\pi\ell}{p})$ to get (1.43). □

The proof above shows that $A(\frac{\ell}{p};n)$ can be written in terms of the sums of Kloosterman sums (2.43) and (2.48). In the following two subsections, we will prove the claim that Bringmann's asymptotic formula (1.41), when summing up to infinity, matches our exact formula (1.43). To be precise, we will show that the Fourier expansion of the $\ell$-th component of $\mathbf{P}_\infty$ matches the first sum in (1.41), and the Fourier expansion of the $\ell$-th component of $\mathbf{P}_0$ matches the second sum on $r$ in (1.41).

### 7.2.1 Contribution from $\mathrm{P}_\infty$

Recall that for a prime $p \geq 5$, a positive integer $c$ such that $p|c$, and $0 < d < c$ such that $(d,c) = 1$, the Dedekind sum $s(d,c)$ is defined in (1.18). As $c$ is always clear in this subsection, we denote $d'$ as $d'_c$ for simplicity, i.e. it is defined by $dd' \equiv -1 \pmod{c}$ if $c$ is odd and $dd' \equiv -1 \pmod{2c}$ if $c$ is even. Also recall the notation that $a$ is given by $0 < a < c$ with $ad \equiv 1 \pmod{c}$ and $[a\ell]$ is defined by $0 \leq [a\ell] < p$ such that $[a\ell] \equiv a\ell \pmod{p}$.

In this subsection we prove (1.44) in the Remark of Theorem 1.14. We conjugate (1.41) since its left side is real and see the first sum is

$$\frac{2\pi e(-\frac{1}{8})}{(24n-1)^{\frac{1}{4}}} \sum_{p|c \leq \sqrt{n}} \frac{e(-\frac{1}{8})\overline{B_{\ell,p,c}(-n,0)}}{c} I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{6c}\right).$$

Then (1.37) gives

$$\overline{B_{\ell,p,c}(-n,0)} = \sum_{d \pmod{c}^*} (-1)^{\ell c+1} \frac{\sin(\frac{\pi\ell}{p})}{\sin(\frac{\pi\ell d'}{p})} \exp\left(-\pi i s(d,c) + \frac{3\pi i \ell^2 c d'}{p^2}\right) e\left(\frac{nd}{c}\right).$$

On the other hand, recall (7.5):

$$S^{(\ell)}_{\infty\infty}(m,n,c,\mu_p) = e(-\tfrac{1}{8}) \sum_{\substack{d \pmod{c}^* \\ ad \equiv 1 \pmod{c}}} \frac{\overline{\mu(c,d,[a\ell],p)}}{\sin(\frac{\pi[a\ell]}{p})} e^{-\pi i s(d,c)} e\left(\frac{ma+nd}{c}\right).$$

To prove (1.44), it suffices to show that for all $d \pmod{c}^*$, we have

$$\frac{(-1)^{\ell c+1}}{\sin(\frac{\pi\ell d'}{p})} \exp\left(\frac{3\pi i c d' \ell^2}{p^2}\right) = \frac{\overline{\mu(c,d,[a\ell],p)}}{\sin(\frac{\pi[a\ell]}{p})}. \tag{7.21}$$

We will show that both sides are equal to

$$\frac{(-1)^{\ell c}}{\sin(\frac{\pi a\ell}{p})} \exp\left(-\frac{3\pi i c a \ell^2}{p^2}\right). \tag{7.22}$$

First we prove that the left side of (7.21) equals (7.22). When $c$ is odd, we write $c = (2k+1)p$ for some integer $k$. Since $dd' \equiv -1 \pmod{c}$, we can pick $d' = c - a$. Then

$$\begin{aligned}
\frac{(-1)^{\ell c+1}}{\sin(\frac{\pi\ell d'}{p})} \exp\left(\frac{3\pi i \ell^2 c d'}{p^2}\right) &= \frac{-(-1)^{\ell}}{\sin(\frac{\pi\ell c}{p} - \frac{\pi\ell a}{p})} \exp\left(\frac{3\pi i \ell^2 c^2}{p^2} - \frac{3\pi i \ell^2 c a}{p^2}\right) \\
&= \frac{-(-1)^{\ell}}{-(-1)^{\ell(2k+1)}\sin(\frac{\pi\ell a}{p})} (-1)^{\ell^2(2k+1)^2} \exp\left(-\frac{3\pi i \ell^2 c a}{p^2}\right) \\
&= \frac{(-1)^{\ell}}{\sin(\frac{\pi\ell a}{p})} \exp\left(-\frac{3\pi i \ell^2 c a}{p^2}\right),
\end{aligned}$$

which equals (7.22). When $c$ is even, we write $c = 2kp$ for some positive integer $k$. We pick $0 < a < 2kp$ for $ad \equiv 1 \pmod{2kp}$ and $0 < d' < 4kp$ for $d'd \equiv -1 \pmod{4kp}$. Observe that (7.22) is the same if we change $a$ to $a \pm 2kp$, so we can pick $a = 2c - d'$ here and a similar process shows that the left side of (7.21) equals

(7.22) when $c$ is even.

Next we prove that the right side of (7.21) equals (7.22). Define the integer $t \geq 0$ by $[a\ell] = a\ell - tp$, $k$ by $c = kp$, and $b$ by $ad = 1 + bc$. By (2.34), we have

$$\begin{aligned}
&\frac{\overline{\mu(c,d,[a\ell],p)}}{\sin(\frac{\pi[a\ell]}{p})} \\
&= \exp\left(-\frac{3\pi i c d(a\ell - tp)^2}{p^2}\right)(-1)^{\frac{c(a\ell-tp)}{p}}(-1)^{\left\lfloor \frac{d(a\ell-tp)}{p}\right\rfloor} \Big/ \sin\left(\frac{\pi a\ell}{p} - \pi t\right) \\
&= \exp\left(-\frac{3\pi i c a\ell^2}{p^2} - \frac{3\pi i c a b c \ell^2}{p^2} - 3\pi i c d t^2\right)(-1)^{a\ell k - tc + \left\lfloor \frac{\ell}{p} + b\ell k - td\right\rfloor + t} \Big/ \sin\left(\frac{\pi a\ell}{p}\right).
\end{aligned}$$

The above formula equals $\exp(-\frac{3\pi i c a\ell^2}{p^2})/\sin(\frac{\pi a\ell}{p})$ times $(-1)$ to the power of

$$\begin{aligned}
&ab\ell^2k^2 - cdt^2 + a\ell k - tc + b\ell k - td + t \\
&\equiv ab\ell k + cdt + a\ell k + tc + b\ell k + td + t \\
&\equiv (a+1)(b+1)\ell k + \ell k + (c+1)(d+1)t \\
&\equiv \ell k \equiv \ell c \pmod 2.
\end{aligned}$$

The last step uses $(x+1)(y+1) \equiv 0 \pmod 2$ whenever $(x,y) = 1$.

*Remark.* From the proof above, for $p|c$ and $0 < [a\ell] = a\ell - tp < p$, we also have

$$\overline{\mu(c,d,[a\ell],p)} = \exp\left(-\frac{3\pi i c a\ell^2}{p^2}\right)(-1)^{\ell c + t}. \tag{7.23}$$

This formula is helpful in Chapter 8.

### 7.2.2 Contribution from $\mathbf{P}_0$

In this subsection we prove (1.45) in the Remark of Theorem 1.14. Recall the definition of $\delta_{\ell,p,a,r}$ in (1.39), of $m_{\ell,p,a,r}$ in (1.40), of $\alpha_0^{(\ell)}$ in (2.36) and of $\mathbf{X}_r$ in (2.40). In Bringmann's asymptotic formula (1.41), the second sum on $r$ (after conjugation) becomes

$$\frac{4\pi \sin(\frac{\pi\ell}{p})}{(n-\frac{1}{24})^{\frac14}} \sum_{r\geq 0} \sum_{\substack{a>0:\, p\nmid a,\\ \delta_{\ell,p,a,r}>0}} \frac{\overline{D_{\ell,p,a,r}(-n, m_{\ell,p,a,r})}}{a\cdot \delta_{\ell,p,a,r}^{-\frac14}} I_{\frac12}\left(\frac{4\pi}{a}\left|\delta_{\ell,p,a,r}\left(n-\frac{1}{24}\right)\right|^{\frac12}\right).$$

Recall (7.9):

$$S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, a, \mu_p; r) = e(-\tfrac18) \sum_{\substack{b:\ b \ (\mathrm{mod}\ a)^*\\ 0<c<pa,\ p|c\\ \text{s.t. } ad-bc=1}} \overline{\mu(c,d,[a\ell],p)} e^{-\pi i s(d,c)} e\left(\frac{-X_{r,0}^{([a\ell])}\frac{c}{p} + n_\infty b}{a} + \frac{a+d}{24c}\right).$$

We denote $b'_a$ as $b'$ for simplicity, i.e. $b'$ is defined by $bb' \equiv -1 \pmod a$ if $a$ is odd and by $bb' \equiv -1 \pmod{2a}$ if $a$ is even. Moreover, we still denote positive integers $t$ by $a\ell - [a\ell] = tp$ and $k$ by $c = kp$.

We have $c \equiv b' \pmod{a}$ for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$. Hence we can rewrite $\overline{D_{\ell,p,a,r}}$ as:

$$\begin{aligned}\overline{D_{\ell,p,a,r}(-n, m_{\ell,p,a,r})} &= (-1)^{a\ell+[a\ell]} \sum_{b \pmod{a}^*} \overline{\omega_{b,a}}\ \overline{e\left(\frac{m_{\ell,p,a,r}b' - nb}{a}\right)} \\ &= (-1)^{a\ell-[a\ell]} \sum_{b \pmod{a}^*} e^{-\pi i s(b,a)} e\left(\frac{-m_{\ell,p,a,r}c + nb}{a}\right).\end{aligned}$$

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p)$, with our choice $c \geq 0$ and $a > 0$, we need the relationship between $e(-\pi i s(b,a))$ and $e^{-\pi i s(d,c)}$. Denote $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Recall $w_{\frac{1}{2}}$ in Definition 1.1. We have

$$w_{\frac{1}{2}}(S,\gamma) = (cz+d)^{\frac{1}{2}} \left(\frac{az+b}{cz+d}\right)^{\frac{1}{2}} (az+b)^{-\frac{1}{2}} = 1$$

because $cz+b$, $\frac{az+b}{cz+d}$ and $az+b$ are in $\mathbb{H}$ for $z \in \mathbb{H}$. Therefore, we have $\nu_\eta(S\gamma) = \nu_\eta(S)\nu_\eta(\gamma)$. With the help of $\nu_\eta(S) = e(-\frac{1}{8})$ by (1.18), we get

$$e^{-\pi i s(b,a)} = e(-\tfrac{1}{8}) e^{-\pi i s(d,c)} e(\tfrac{a+d}{24c} + \tfrac{c-b}{24a}).$$

Then we continue:

$$\begin{aligned}\overline{D_{\ell,p,a,r}(-n, m_{\ell,p,a,r})} &= (-1)^{tp} e(-\tfrac{1}{8}) \sum_{b \pmod{a}^*} \overline{\omega_{d,c}}\ e\left(\frac{-m_{\ell,p,a,r}c + nb}{a} + \frac{a+d}{24c} + \frac{c-b}{24a}\right) \\ &= (-1)^t e(-\tfrac{1}{8}) \sum_{b \pmod{a}^*} \overline{\omega_{d,c}}\ e\left(\frac{a+d}{24c}\right) e\left(\frac{(\frac{1}{24} - m_{\ell,p,a,r})c + (n - \frac{1}{24})b}{a}\right).\end{aligned}$$

Compare with the formula of $S_{0\infty}^{(\ell)}(X_r^{([a\ell])}, n, c, \mu_p)$ where $X_{r,0}^{([a\ell])} = -p\delta_{\ell,p,a,r}$, we are left to prove

$$\overline{\mu(c,d,[a\ell],p)}\ e\left(\frac{\delta_{\ell,p,a,r}c}{a}\right) = (-1)^t e\left(\frac{(\frac{1}{24} - m_{\ell,p,a,r})c}{a}\right). \tag{7.24}$$

By (7.23), when $0 < \frac{[a\ell]}{p} < \frac{1}{6}$, recalling $[a\ell] = a\ell - tp$ and $c = kp$, we have

$$\begin{aligned}&\overline{\mu(c,d,[a\ell],p)}\ e\left(\frac{(\delta_{\ell,p,a,r} - \frac{1}{24})c}{a}\right) \\ &= (-1)^{\ell c+t} e\left(-\frac{3ca\ell^2}{2p^2} - \frac{c(1+2r)(a\ell-tp)}{2ap} + \frac{3c(a\ell-tp)^2}{2ap^2}\right) \\ &= (-1)^{\ell k+t} e\left(-\frac{3ca\ell^2}{2p^2} - \frac{k\ell(1+2r)}{2} + \frac{ct(1+2r)}{2a} + \frac{3ca\ell^2}{2p^2} - \frac{3c\ell t}{p} + \frac{3ct^2}{2a}\right) \\ &= (-1)^t e\left(\frac{c}{a}\left(\frac{(1+2r)t}{2} + \frac{3}{2}t^2\right)\right).\end{aligned}$$

On the other hand, by (1.40),

$$-m_{\ell,p,a,r} = \frac{1}{2p^2}\Big(3(a\ell - [a\ell])^2 + p(1+2r)(a\ell - [a\ell])\Big) = \frac{3}{2}t^2 + \frac{(1+2r)t}{2}.$$

This gives (7.24). The proof when $\frac{5}{6} < \frac{[a\ell]}{p} < 1$ is similar: we have

$$\begin{aligned}
&\overline{\mu(c,d,[a\ell],p)}\ e\left(\frac{(\delta_{\ell,p,a,r}-\frac{1}{24})c}{a}\right)\\
&=(-1)^{\ell c+t}e\left(-\frac{3ca\ell^2}{2p^2}-\frac{5c(a\ell-tp)}{2ap}+\frac{3c(a\ell-tp)^2}{2ap^2}+(1-r)\frac{c}{a}+\frac{cr(a\ell-tp)}{ap}\right)\\
&=(-1)^{\ell k+t}e\left(-\frac{5k\ell}{2}+\frac{5ct}{2a}-3k\ell t+\frac{3ct^2}{2a}+(1-r)\frac{c}{a}+rk\ell-rt\frac{c}{a}\right)\\
&=(-1)^{t}e\left(\frac{c}{a}\left(\frac{(5-2r)t}{2}+\frac{3}{2}t^2+1-r\right)\right).
\end{aligned}$$

When $\frac{5}{6} < \frac{[a\ell]}{p} < 1$, by (1.40) we also have

$$-m_{\ell,p,a,r}=\frac{3}{2}t^2+\frac{(5-2r)t}{2}+1-r.$$

Now we still get (7.24) and (1.45) follows.

*Remark.* From the proof we can rewrite

$$S_{0\infty}^{(\ell)}\left(\lceil -p\delta_{\ell,p,a,r}\rceil, n, a, \mu\right)=(-1)^{a\ell-[a\ell]}\sum_{b \pmod{a}^*} e^{-\pi i s(b,a)}e\left(\frac{-m_{\ell,p,a,r}c+nb}{a}\right) \tag{7.25}$$

where $0<[a\ell]=a\ell-tp<p$ and

$$-m_{\ell,p,a,r}=\begin{cases}\frac{3}{2}t^2+\frac{1+2r}{2}t, & \text{when } 0<\frac{[a\ell]}{p}<\frac{1}{6},\\ \frac{3}{2}t^2+\frac{5-2r}{2}t+1-r, & \text{when } \frac{5}{6}<\frac{[a\ell]}{p}<1.\end{cases}$$

This expression is helpful in the proof of Theorem 1.15 in Chapter 8 for the case $p=7$.

# Chapter 8

# Equidistribution of ranks modulo 5 and 7

In this chapter we prove Theorem 1.15, where Corollary 1.16 is a direct result once we have the exact formula (Theorem 1.14). This proof is length and consists of checking many cases. However, the proof only uses basic properties of congruences and Kronecker symbols. There are many tables recording data of arguments for complex numbers among the proof.

## 8.1 Proof of Theorem 1.15, claim (1)

In this section we prove claim (1) of Theorem 1.15, which is for the case $p = 5$. In Proposition 7.1 we find

$$S^{(\ell)}_{\infty\infty}(0, 5n+4, c, \mu_p) = e(-\tfrac{1}{8}) \sum_{\substack{d \pmod{c}^* \\ ad\equiv 1 \pmod{c}}} \frac{\overline{\mu(c,d,[a\ell],5)}}{\sin(\frac{\pi[a\ell]}{5})} e^{-\pi i s(d,c)} e\left(\frac{(5n+4)d}{c}\right).$$

We only consider $\ell = 1, 2$ because $A(\frac{\ell}{p}; n) = A(1 - \frac{\ell}{p}; n)$.

Denote $c'$ by $c = 5c'$. For $r \pmod{c'}^*$, we define

$$V(r, c) := \{d \pmod{c}^* : \ d \equiv r \pmod{c'}\}.$$

For example, $V(1, 30) = \{d \pmod{30}^* : d \equiv 1, 7, 13, 19 \pmod{30}\}$ and $V(4, 25) = \{d \pmod{25}^* : d \equiv 4, 9, 14, 19, 24 \pmod{25}\}$. We will not restrict $0 < d < c$ in $V(r, c)$ because changing $d$ to $d + c$ will not affect the value of our Kloosterman sums. Clearly, $|V(r, c)| = 4$ if $5 \| c$ and $|V(r, c)| = 5$ if $25 | c$. Moreover, $(\mathbb{Z}/c\mathbb{Z})^*$ is the disjoint union

$$(\mathbb{Z}/c\mathbb{Z})^* = \bigcup_{r \pmod{c'}^*} V(r, c), \quad \text{where} \quad V(r_1, c) \cap V(r_2, c) = \emptyset \ \text{ if } r_1 \not\equiv r_2 \pmod{c'}.$$

From (7.21) and (7.22) where $p = 5$, we have

$$\frac{\overline{\mu(c,d,[a\ell],5)}}{\sin(\frac{\pi[a\ell]}{5})} = \frac{(-1)^{\ell c}}{\sin(\frac{\pi a\ell}{5})} \exp\left(-\frac{3\pi i c' a \ell^2}{5}\right).$$

We claim that for $\ell = 1, 2$, the sum on $V(r,c)$ satisfies

$$s_{r,c} := \sum_{d \in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{10}\right)}{\sin(\frac{\pi a \ell}{5})} e^{-\pi i s(d,c)} e\left(\frac{4d}{c}\right) = 0. \tag{8.1}$$

If (8.1) is true, then

$$S^{(\ell)}_{\infty\infty}(0, 5n+4, c, \mu_p) = e(-\tfrac{1}{8})(-1)^{\ell c} \sum_{r \pmod{c'})^*} s_{r,c} e\left(\frac{nr}{c'}\right)(-1)^{\ell c} = 0$$

for all $n \in \mathbb{Z}$, $\ell = 1, 2$, and we have proved Theorem 1.15 in the case $p = 5$.

In the following subsections §7.1-§7.4, we prove (8.1) when $5\|c$. In §7.5, we prove (8.1) when $25|c$. Suppose now that $5\|c$. Since $|V(r,c)| = 4$, let $\beta \in \{1,2,3,4\}$ such that $\beta c' \equiv 1 \pmod 5$ and we make a special choice of $V(r,c)$ as

$$V(r,c) = \{d_1, d_2, d_3, d_4\} \quad \text{where } d_j \equiv j \pmod 5 \text{ and } d_{j+1} = d_1 + j\beta c'. \tag{8.2}$$

We also take $a_j$ such that $a_j \equiv j \pmod 5$, $a_{j+1} = a_1 + j\beta c'$, and

$$a_{\overline{j_{\{5\}}}} d_j \equiv 1 \pmod c \quad \text{because } \overline{j_{\{5\}}} \cdot j \equiv 1 \pmod 5. \tag{8.3}$$

These choices do not affect the sum (8.1) because $s_{r,c}$ has period $c$ in both $a$ and $d$. In (8.1), we denote each single summation term as

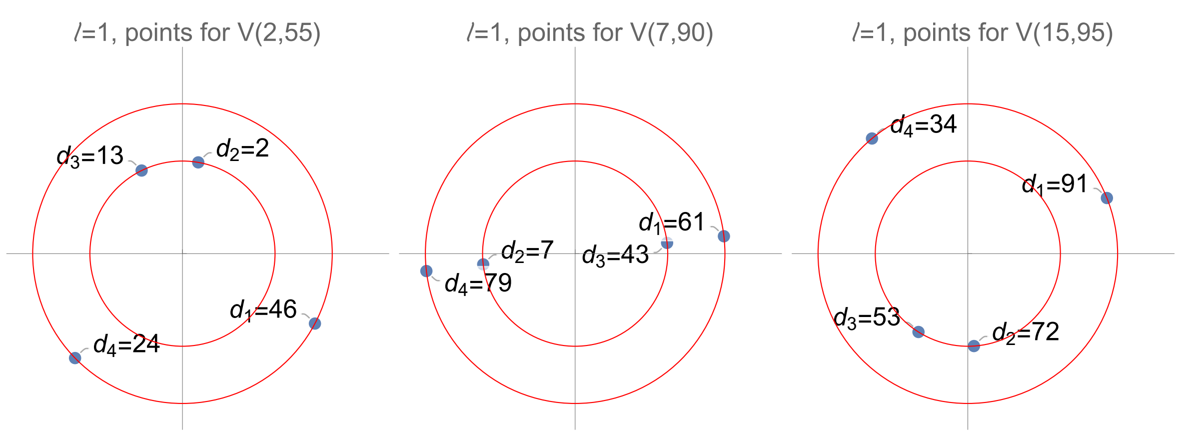
$$P(d) := \frac{e\left(-\frac{3c'a\ell^2}{10}\right)}{\sin(\frac{\pi a \ell}{5})} \cdot e\left(-\frac{12cs(d,c)}{24c}\right) \cdot e\left(\frac{4d}{c}\right) =: P_1(d) \cdot P_2(d) \cdot P_3(d) \tag{8.4}$$

where $P_1(d) := e(-\frac{3c'a\ell^2}{10})/\sin(\frac{\pi a \ell}{5})$, $P_2 := \exp(-\pi i s(d,c))$, and $P_3 := e(\frac{4d}{c})$.

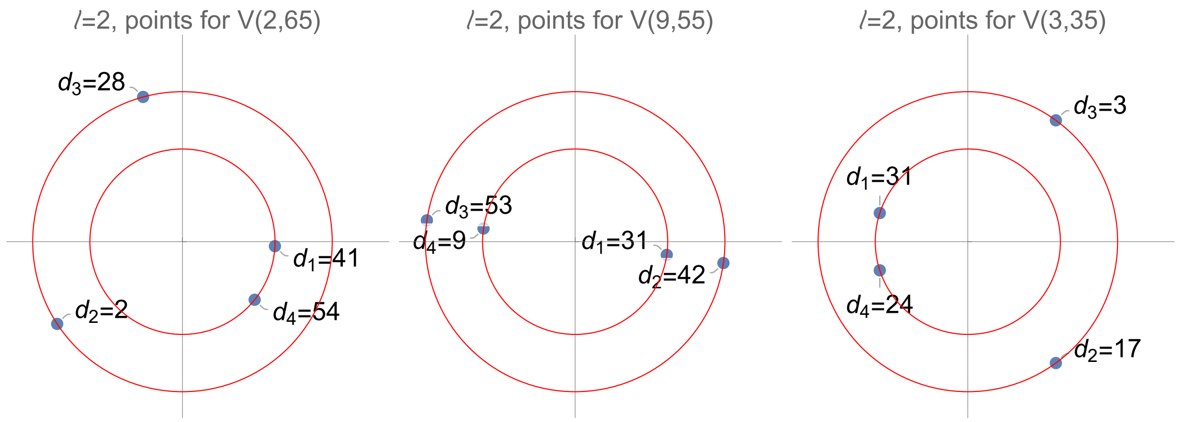
*Remark.* We keep $24c$ at the denominator of $P_2(d)$ because the congruence properties of the Dedekind sum are of the form $12cs(d,c)$. See (8.8)-(8.11) for details.

We claim that the set of points $P(d)$ for $d \in V(r,c)$ must have the relative position as one of the following six configurations. Here $0 < d_j < c$ for simplicity but we use (8.2) in the proof.

- $\ell = 1$:




- $\ell = 2$:




Here we explain the styles. Each graph above has two circles with inner one of radius $\csc(\frac{2\pi}{5})$ and outer one with radius $\csc(\frac{\pi}{5})$. When $\ell = 1$, the value of $P(d_1)$ and $P(d_4)$ will be on the outer circle ($P(d_2)$ and $P(d_3)$ on the inner circle) because the term $P_1(d_j)$ has denominator $\sin(\frac{\pi a_j \ell}{5})$. When $\ell = 2$, $P(d_1)$ and $P(d_4)$ will be on the inner circle.

We describe the relative argument differences via the following notations. Denote

$$\mathrm{Arg}_j(d_u \to d_v; \ell), \quad \text{for } j \in \{1,2,3\},\ u,v \in \{1,2,3,4\},\ \text{and } \ell \in \{1,2\} \tag{8.5}$$

be the argument difference (as the proportion of $2\pi$, positive when going counter-clockwise) contributed from $P_j$ going from $d_u$ to $d_v$ when $\ell \in \{1,2\}$. To be precise, if we denote $P_j(d_u) = R_{j,u}\exp(i\Theta_{j,u})$ for $R_{j,u}, \Theta_{j,u} \in \mathbb{R}$, then

$$\mathrm{Arg}_j(d_u \to d_v; \ell) = \alpha \quad \Leftrightarrow \quad \Theta_{j,v} - \Theta_{j,u} = \alpha \cdot 2\pi + 2k\pi \text{ for some } k \in \mathbb{Z}.$$

We say two argument differences equal: $\mathrm{Arg}_j(d_u \to d_v; \ell) = \mathrm{Arg}_j(d_w \to d_x; \ell)$ if their difference is an integer.

Although the $P_2$ and $P_3$ terms are not affected by the value of $\ell$ in (8.4), we still use the notation $\mathrm{Arg}_2(d_u \to d_v; \ell)$ to indicate the different cases for $\ell$. Moreover, we define

$$\mathrm{Arg}(d_u \to d_v; \ell) := \sum_{j=1}^{3} \mathrm{Arg}_j(d_u \to d_v; \ell) \tag{8.6}$$

as the argument difference in total.

The following condition makes the sum of four points $P(d)$ for $d \in V(r,c)$ on each graph to be zero:

**Condition 8.1.** *We have the following six styles for the relative position of these four points.*

- $\ell = 1$. *First graph style: the arguments (as a proportion of $2\pi$) going $d_1 \to d_2 \to d_3 \to d_4 \to d_1$ are $\frac{3}{10}$, $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{3}{10}$, respectively. The second graph style is that all the argument differences are $\frac{1}{2}$, while the third graph style has the reversed order of rotation compared with the first one.*

| $\mathrm{Arg}(d_u \to d_v; 1) \searrow$ | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 1 \pmod 5$ | | $\frac{3}{10}$ | | $\frac{1}{10}$ | | $\frac{3}{10}$ | | $\frac{3}{10}$ | |
| $c' \equiv 2,3 \pmod 5$ | | $\frac{1}{2}$ | | $\frac{1}{2}$ | | $\frac{1}{2}$ | | $\frac{1}{2}$ | |
| $c' \equiv 4 \pmod 5$ | | $-\frac{3}{10}$ | | $-\frac{1}{10}$ | | $-\frac{3}{10}$ | | $-\frac{3}{10}$ | |

- $\ell = 2$. *Here are the styles for the fourth, fifth and sixth graphs.*

| $\mathrm{Arg}(d_u \to d_v; 2) \searrow$ | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 3 \pmod 5$ | | $-\frac{2}{5}$ | | $-\frac{3}{10}$ | | $-\frac{2}{5}$ | | $\frac{1}{10}$ | |
| $c' \equiv 1, 4 \pmod 5$ | | $0$ | | $\frac{1}{2}$ | | $0$ | | $\frac{1}{2}$ | |
| $c' \equiv 2 \pmod 5$ | | $\frac{2}{5}$ | | $\frac{3}{10}$ | | $\frac{2}{5}$ | | $-\frac{1}{10}$ | |

One can check that, whenever the four points on $\mathbb{C}$ satisfy any of the above cases of relative argument differences and corresponding radii, the sum of them becomes 0. This can be explained by

$$\frac{\cos(\frac{\pi}{10})}{\sin(\frac{2\pi}{5})} = \frac{\cos(\frac{3\pi}{10})}{\sin(\frac{\pi}{5})} = 1, \quad \text{where } \frac{1}{\sin(\frac{\pi}{5})} \text{ and } \frac{1}{\sin(\frac{2\pi}{5})} \text{ are the radii.}$$

In other words, we prove (8.1) by showing that for $5\|c$ and every $r \pmod{c'}^*$, the four terms in (8.1) has one of the styles in Condition 8.1.

Before we divide into the cases, we first claim the following lemma:

**Lemma 8.2.** *For $\ell \in \{1, 2\}$, we have*

$$\mathrm{Arg}(d_1 \to d_2; \ell) + \mathrm{Arg}(d_4 \to d_3; \ell) = 0 \quad \textit{and} \quad \mathrm{Arg}(d_1 \to d_3; \ell) + \mathrm{Arg}(d_4 \to d_2; \ell) = 0. \tag{8.7}$$

Granted the above reduction, to prove that each case of the argument differences are one of the cases in Condition 8.1, we only need to verify that

$$\mathrm{Arg}(d_1 \to d_4; \ell) \quad \text{and} \quad \mathrm{Arg}(d_1 \to d_2; \ell) \quad \text{for } \ell = 1, 2$$

satisfy Condition 8.1. We prove this by enumerating all the cases. We can list the argument differences for $\mathrm{Arg}_1$ and $\mathrm{Arg}_3$, but for $\mathrm{Arg}_2$, we require the following congruence properties of the Dedekind sum from [60, (4.2)-(4.5)]. Here $(d, c) = 1$ and $\overline{d_{\{m\}}}$ is the inverse of $d \pmod m$ (see Notation 2.9):

$$2\theta cs(d, c) \in \mathbb{Z}, \quad \text{where } \theta = \gcd(c, 3), \tag{8.8}$$
$$12cs(d, c) \equiv d + \overline{d_{\{\theta c\}}} \pmod{\theta c}, \tag{8.9}$$
$$12cs(d, c) \equiv c + 1 - 2\left(\tfrac{d}{c}\right) \pmod 8, \quad \text{if } c \text{ is odd}, \tag{8.10}$$
$$12cs(d, c) \equiv d + \left(c^2 + 3c + 1 + 2c\left(\tfrac{c}{d}\right)\right) \overline{d_{\{8\times 2^\lambda\}}} \pmod{8 \times 2^\lambda}, \quad \text{if } 2^\lambda \| c \text{ for } \lambda \geq 1. \tag{8.11}$$

These congruences determine $12cs(d, c) \pmod{24c}$ uniquely in all the cases ($2|c$ or $2 \nmid c$, $3|c$ or $3 \nmid c$), which is the reason why we keep $24c$ in the denominator of $P_2(d)$.

*Proof of Lemma 8.2.* Note that

$$\mathrm{Arg}(d_u \to d_v; \ell) = \mathrm{Arg}(d_u \to d_w; \ell) + \mathrm{Arg}(d_w \to d_v; \ell)$$

for all $u, v, w \in \{1, 2, 3, 4\}$. Then it suffices to prove

$$\mathrm{Arg}(d_1 \to d_2; \ell) = \mathrm{Arg}(d_3 \to d_4; \ell).$$

Recall our notation for $d_j$ and $a_j$ in (8.2). Since $a_3 - a_1 = a_4 - a_2 = 2\beta c'$, one can show $\mathrm{Arg}_1(d_1 \to d_2; \ell) = \mathrm{Arg}_1(d_3 \to d_4; \ell)$ by

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_3 \ell}{5})/\sin(\tfrac{\pi a_1 \ell}{5})\right) = \mathrm{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5})/\sin(\tfrac{\pi a_2 \ell}{5})\right) = 1.$$

It is also easy to show $\mathrm{Arg}_3(d_1 \to d_2; \ell) = \mathrm{Arg}_3(d_3 \to d_4; \ell)$. For $\mathrm{Arg}_2$, we apply (8.9), (8.10) and (8.11) with the Chinese Remainder Theorem to show that

$$12cs(d_2, c) - 12cs(d_1, c) \equiv 12cs(d_4, c) - 12cs(d_3, c) \text{ for all the corresponding congruences.}$$

When $\gcd(c, 3) = 1$, we have

$$\begin{aligned} 12cs(d_2, c) - 12cs(d_1, c) &\equiv d_2 + a_3 - d_1 - a_1 \equiv 3\beta c' \pmod{c}, \\ 12cs(d_4, c) - 12cs(d_3, c) &\equiv d_4 + a_4 - d_3 - a_2 \equiv 3\beta c' \pmod{c}, \\ 12cs(d_2, c) - 12cs(d_1, c) &\equiv 12cs(d_4, c) - 12cs(d_3, c) \equiv 0 \pmod{6}. \end{aligned}$$

When $3|c$, we apply the congruence

$$\overline{(x+y)_{\{m\}}} - \overline{x_{\{m\}}} \equiv -y\overline{(x+y)_{\{m\}}} \cdot \overline{x_{\{m\}}} \pmod{m} \tag{8.12}$$

to compute

$$\begin{aligned} d_2 + \overline{d_{2\{3c\}}} - d_1 - \overline{d_{1\{3c\}}} &\equiv \beta c'(1 - \overline{d_{2\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}, \\ d_4 + \overline{d_{4\{3c\}}} - d_3 - \overline{d_{3\{3c\}}} &\equiv \beta c'(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{3\{3c\}}}) \pmod{3c}, \end{aligned}$$

which imply

$$12cs(d_2, c) - 12cs(d_1, c) \equiv 12cs(d_4, c) - 12cs(d_3, c) \equiv 0 \pmod{c'}.$$

by (8.9). After dividing by $c'$ (recall that the denominator of $P_2(d)$ is $24c$), we have

$$\begin{aligned} 60s(d_2, c) - 60s(d_1, c) &\equiv \beta(1 - \overline{d_{2\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \equiv \beta(1 - a_3 a_1) \pmod{15}, \\ 60s(d_4, c) - 60s(d_3, c) &\equiv \beta(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{3\{3c\}}}) \equiv \beta(1 - a_4 a_2) \pmod{15}, \end{aligned}$$

because of (8.3) and $\overline{x_{\{un\}}} \equiv \overline{x_{\{vn\}}} \pmod{n}$. Since $a_3 \equiv a_1 \pmod{3}$ and $a_4 \equiv a_2 \pmod{3}$, we have $a_3 a_1 \equiv a_4 a_2 \equiv 1 \pmod{3}$. Moreover, $a_3 a_1 \equiv a_4 a_2 \equiv 3 \pmod{5}$. Hence $a_3 a_1 \equiv a_4 a_2 \equiv 13 \pmod{15}$ and we get

$$60s(d_2, c) - 60s(d_1, c) \equiv 60s(d_4, c) - 60s(d_3, c) \equiv 3\beta \pmod{15}.$$

When $c$ is odd, by (8.10) and $d_{j_1} \equiv d_{j_2} \pmod{c'}$, we have

$$\begin{aligned} 12cs(d_2, c) - 12cs(d_1, c) &\equiv 2(\tfrac{d_1}{c}) - 2(\tfrac{d_1}{c}) \equiv 2(\tfrac{1}{5})(\tfrac{d_1}{c'}) - 2(\tfrac{2}{5})(\tfrac{d_2}{c'}) \equiv 4 \pmod{8}, \\ 12cs(d_4, c) - 12cs(d_3, c) &\equiv 2(\tfrac{d_4}{c}) - 2(\tfrac{d_3}{c}) \equiv 2(\tfrac{4}{5})(\tfrac{d_4}{c'}) - 2(\tfrac{3}{5})(\tfrac{d_3}{c'}) \equiv 4 \pmod{8}. \end{aligned}$$

When $c$ is even and $2^\lambda \| c$ for $\lambda \geq 1$, by (8.11) we have

$$\begin{aligned}
12cs(d_2,c) - 12cs(d_1,c) &\equiv d_2 + (c^2+3c+1)\overline{d_{2\{8\times 2^\lambda\}}} + 2c(\tfrac{c}{d_2})\overline{d_{2\{8\times 2^\lambda\}}} \\
&\quad - d_1 + (c^2+3c+1)\overline{d_{1\{8\times 2^\lambda\}}} + 2c(\tfrac{c}{d_1})\overline{d_{1\{8\times 2^\lambda\}}} \\
&\equiv \beta c'(1 - \overline{d_{2\{8\times 2^\lambda\}}} \cdot \overline{d_{1\{8\times 2^\lambda\}}}) \\
&\quad + 2c(\tfrac{c}{d_2})\overline{d_{2\{8\times 2^\lambda\}}} - 2c(\tfrac{c}{d_1})\overline{d_{1\{8\times 2^\lambda\}}} \pmod{8 \times 2^\lambda}
\end{aligned}$$

Hence above the difference is a multiple of $c'$. Dividing $c'$ and by $x^2 \equiv 1 \pmod 8$ for odd $x$ we have

$$60s(d_2,c) - 60s(d_1,c) \equiv \beta(1 - (c^2+3c+1)d_2d_1) + 2(\tfrac{c}{d_2})d_2 - 2(\tfrac{c}{d_1})d_1 \pmod 8.$$

Similarly, we also have

$$60s(d_4,c) - 60s(d_3,c) \equiv \beta(1 - (c^2+3c+1)d_4d_3) + 2(\tfrac{c}{d_4})d_4 - 2(\tfrac{c}{d_3})d_3 \pmod 8.$$

Dividing into cases for $4|c$ or $2\|c$ with $c' \equiv 2$ or $6 \pmod 8$, one can conclude

$$d_2d_1 \equiv d_4d_3 \pmod 8.$$

For the remaining part, we only need to determine $(\frac{c}{d_j})d_j \equiv \pm 1 \pmod 4$ for $j \in \{1,2,3,4\}$. Since $d_3 \equiv d_1 \pmod 4$ and $d_2 \equiv d_4 \pmod 4$, it is not hard to show that

$$(\tfrac{c}{d_2})d_2 - (\tfrac{c}{d_1})d_1 \equiv (\tfrac{c}{d_4})d_4 - (\tfrac{c}{d_3})d_3 \pmod 4.$$

Combining all the congruence equations in this proof, we have shown that

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \mathrm{Arg}_2(d_3 \to d_4; \ell) \quad \text{for } \ell \in \{1,2\}$$

by proving

$$\frac{12cs(d_2,c) - 12cs(d_1,c)}{24c} - \frac{12cs(d_4,c) - 12cs(d_3,c)}{24c} \in \mathbb{Z}$$

in all the cases for $c$ ($2|c$ or $2 \nmid c$, $3|c$ or $3 \nmid c$). The lemma follows.

□

### 8.1.1 $2 \nmid c'$, $3 \nmid c'$, and $5 \nmid c'$

We first deal with the case for $c' \equiv 1, 7, 11, 13, 17, 19, 23, 29 \pmod{30}$. Recall our notations

$$d_4 = d_1 + 3\beta c', \quad d_2 = d_1 + \beta c', \quad a_4 = a_1 + 3\beta c', \quad a_3 = a_1 + 2\beta c', \quad \beta c' \equiv 1 \pmod 5.$$

The argument differences $\mathrm{Arg}_j(d_1 \to d_4; \ell)$ for $j = 1,2,3$ are given by the arguments of

$$\frac{e\left(-\frac{9}{10}\beta c'^2 \ell^2\right)}{\mathrm{sgn}\left(\sin(\frac{\pi a_4 \ell}{5}) / \sin(\frac{\pi a_1 \ell}{5})\right)}, \quad e\left(-\frac{12cs(d_4,c) - 12cs(d_1,c)}{24c}\right), \quad \text{and } e\left(\frac{2\beta}{5}\right),$$

respectively. First we have

$$\operatorname{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5})/\sin(\tfrac{\pi a_1 \ell}{5})\right) = -1 \quad \text{whenever } \begin{cases} \ell = 1 \\ 3\beta c' \equiv 8 \pmod{10} \end{cases} \text{ or } \ell = 2. \tag{8.13}$$

This is easy to prove because $3\beta c' \times 2 \equiv 6 \pmod{10}$.

By (8.9), we have $\theta = 1$ and

$$-12cs(d_4, c) + 12cs(d_1, c) \equiv -d_4 - a_4 + d_1 + a_1 \equiv -6\beta c' \equiv -\beta c' \pmod{c}. \tag{8.14}$$

Moreover, we have $-12cs(d_4, c) + 12cs(d_1, c) \equiv 0 \pmod 6$ and

$$-12cs(d_4,c) + 12cs(d_1,c) \equiv 2\left(\left(\tfrac{d_4}{c}\right) - \left(\tfrac{d_1}{c}\right)\right) \equiv 2\left(\left(\tfrac{d_4}{5}\right)\left(\tfrac{d_4}{c'}\right) - \left(\tfrac{d_1}{5}\right)\left(\tfrac{d_1}{c'}\right)\right) \equiv 0 \pmod 8.$$

Here we have used $\left(\frac{d_j}{5}\right) = 1$ for $j = 1, 4$ and $d_j \equiv d_1 \pmod{c'}$ for all $j$. Then,

$$-12cs(d_4, c) + 12cs(d_1, c) \equiv 0 \pmod{24}. \tag{8.15}$$

Combining (8.14) and (8.15), since $c'$ is odd, we can divide $24c'$ on both the denominator and numerator in $P_2$. By $24^{-1} = 4$, we get

$$\operatorname{Arg}_2(d_1 \to d_4; \ell) = \frac{\beta}{5}.$$

Now we have Table 8.1. In the row of $\operatorname{Arg}_1(d_1 \to d_4; 1)$, we see $+\frac{1}{2}$ because the sign difference $\operatorname{sgn}\left(\sin(\frac{\pi a_4 \ell}{5})/\sin(\frac{\pi a_1 \ell}{5})\right) = -1$ when $3\beta c' \equiv 8 \pmod{10}$. The $\operatorname{Arg}_1(d_1 \to d_4; 2)$ always need $+\frac{1}{2}$ because $3\beta c' \times 2 \equiv 6 \pmod{10}$. The upper-half table is for the case $\ell = 1$ and the lower-half table is for $\ell = 2$.

| $c' \pmod{30}$ | 1 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
|---|---|---|---|---|---|---|---|---|
| $\beta$ | 1 | 3 | 1 | 2 | 3 | 4 | 2 | 4 |
| $3\beta c' \pmod{10}$ | 3 | 3 | 3 | $\boxed{8}$ | 3 | $\boxed{8}$ | $\boxed{8}$ | $\boxed{8}$ |
| $-9\beta c'^2 \pmod{10}$ | 1 | 7 | 1 | 8 | 7 | 4 | 7 | 4 |
| $\operatorname{Arg}_1(d_1 \to d_4; 1)$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{2}{10}+\frac{1}{2}$ | $-\frac{3}{10}$ | $-\frac{6}{10}+\frac{1}{2}$ | $-\frac{3}{10}+\frac{1}{2}$ | $-\frac{6}{10}+\frac{1}{2}$ |
| $\operatorname{Arg}_2(d_1 \to d_4; 1)$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ |
| $\operatorname{Arg}_3(d_1 \to d_4; 1)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ |
| Total $\operatorname{Arg}(d_1 \to d_4; 1)$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{3}{10}$ |
| $3\beta c' \pmod{10}$ | 3 | 3 | 3 | 8 | 3 | 8 | 8 | 8 |
| $-18\beta c'^2 \equiv 2c' \pmod 5$ | 2 | 4 | 2 | 1 | 4 | 3 | 1 | 3 |
| $\operatorname{Arg}_1(d_1 \to d_4; 2) : \frac{1}{2} + \frac{2c'}{5}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ |
| $\operatorname{Arg}_2(d_1 \to d_4; 2)$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ |
| $\operatorname{Arg}_3(d_1 \to d_4; 2)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ |
| Total $\operatorname{Arg}(d_1 \to d_4; 2)$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $-\frac{1}{10}$ | $\frac{1}{2}$ |

Table 8.1: Table for $\operatorname{Arg}(d_1 \to d_4; \ell)$; $2 \nmid c$, $3 \nmid c$, $5 \nmid c$.

For $\operatorname{Arg}_j(d_1 \to d_2; \ell)$, recall $a_3 d_2 \equiv 1 \pmod c$. The argument differences $\operatorname{Arg}_j(d_1 \to d_2; \ell)$ for $j = 1, 2, 3$

are given by

$$\frac{e\left(-\frac{3}{5}\beta c'^2\ell^2\right)}{\operatorname{sgn}\left(\sin(\frac{\pi a_3\ell}{5})/\sin(\frac{\pi a_1\ell}{5})\right)}, \quad e\left(-\frac{12cs(d_2,c)-12cs(d_1,c)}{24c}\right), \quad e\left(\frac{4\beta}{5}\right),$$

respectively. Since $2\beta c'\ell \equiv 2\ell \pmod{10}$, we always have

$$\operatorname{sgn}\left(\sin(\tfrac{\pi a_3}{5})/\sin(\tfrac{\pi a_1}{5})\right)=1 \quad \text{and} \quad \operatorname{sgn}\left(\sin(\tfrac{2\pi a_3}{5})/\sin(\tfrac{2\pi a_1}{5})\right)=-1. \tag{8.16}$$

Moreover, from (8.9) we have

$$12cs(d_2,c)-12cs(d_1,c)\equiv d_2+a_3-d_1-a_1\equiv 3\beta c' \pmod{c}, \tag{8.17}$$

$$12cs(d_2,c)-12cs(d_1,c)\equiv -2\left(-(\tfrac{d_2}{c'})-(\tfrac{d_1}{c'})\right)\equiv 4 \pmod{8},$$

$12cs(d_2,c)-12cs(d_1,c)\equiv 0 \pmod{6}$, and

$$12cs(d_2,c)-12cs(d_1,c)\equiv 12 \pmod{24}. \tag{8.18}$$

Combining (8.17) and (8.18), we divide by $c'$ and determine the unique value for

$$-(60s(d_2,c)-60s(d_1,c)) \quad \text{congruent to} \quad -3\beta \pmod{5} \quad \text{and} \quad 12 \pmod{24}$$

modulo 120. This gives the contribution of the argument difference from $P_2$. Now we can make Table 8.2.

| $c' \pmod{30}$ | 1 | 7 | 11 | 13 | 17 | 19 | 23 | 29 |
|---|---|---|---|---|---|---|---|---|
| $\beta$ | 1 | 3 | 1 | 2 | 3 | 4 | 2 | 4 |
| $-3c' \pmod 5$ | 2 | 4 | 2 | 1 | 4 | 3 | 1 | 3 |
| $\mathrm{Arg}_1(d_1\to d_2;1)$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ |
| $\mathrm{Arg}_2(d_1\to d_2;1)$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_3(d_1\to d_2;1)$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1\to d_2;1)$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ |
| $-3c'\times 4 \pmod 5$ | 3 | 1 | 3 | 4 | 1 | 2 | 4 | 2 |
| $\mathrm{Arg}_1(d_1\to d_2;2):\frac{1}{2}-\frac{12c'}{5}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_2(d_1\to d_2;2)$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_3(d_1\to d_2;2)$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1\to d_2;2)$ | 0 | $\frac{2}{5}$ | 0 | $-\frac{2}{5}$ | $\frac{2}{5}$ | 0 | $-\frac{2}{5}$ | 0 |

Table 8.2: Table for $\mathrm{Arg}(d_1\to d_2;\ell)$; $2\nmid c$, $3\nmid c$, $5\nmid c$.

Combining Table 8.1 and Table 8.2, we see that $\mathrm{Arg}(d_1\to d_4;\ell)$ and $\mathrm{Arg}(d_1\to d_2;\ell)$ for $\ell=1,2$ satisfy the styles in Condition 8.1. This finishes the proof when $2\nmid c'$, $3\nmid c'$ and $5\nmid c'$.

### 8.1.2 $2\nmid c'$, $3|c'$, and $5\nmid c'$

These are the cases for $c'\equiv 3,9,21,27 \pmod{30}$. For $\mathrm{Arg}_1(d_1\to d_4;\ell)$ we use (8.13):

$$\operatorname{sgn}\left(\sin(\tfrac{\pi a_4\ell}{5})/\sin(\tfrac{\pi a_1\ell}{5})\right)=-1 \quad \text{whenever} \begin{cases}\ell=1\\ 3\beta c'\equiv 8 \pmod{10}\end{cases} \quad \text{or } \ell=2.$$

For $\mathrm{Arg}_2(d_1 \to d_4;\ell)$, we need the congruence equality (8.12). By (8.9), we have

$$12cs(d_4,c) - 12cs(d_1,c) \equiv d_4 + \overline{d_{4\{3c\}}} - d_1 - \overline{d_{1\{3c\}}} \equiv 3\beta c'(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{8.19}$$

By (8.10) we also have

$$12cs(d_4,c) - 12cs(d_1,c) \equiv 0 \pmod 8. \tag{8.20}$$

Dividing the numerator and denominator of $P_2$ by $24c'$, we observe that

$$-\tfrac{5}{2}(s(d_4,c) - s(d_1,c)) \equiv -\overline{8_{\{5\}}}\beta(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \equiv \beta \pmod 5 \tag{8.21}$$

because $\overline{d_{j\{3c\}}} \equiv \overline{d_{j\{5\}}} \equiv j \pmod 5$ for $j = 1, 4$. Now we get $\mathrm{Arg}_2(d_1 \to d_4;\ell) = \frac{\beta}{5}$. Since $\mathrm{Arg}_3(d_1 \to d_4;\ell) = \frac{2\beta}{5}$, we have Table 8.3.

| $c' \pmod{30}$ | 3 | 9 | 21 | 27 |
|---|---|---|---|---|
| $\beta$ | 2 | 4 | 1 | 3 |
| $3\beta c' \pmod{10}$ | 8 | 8 | 3 | 3 |
| $-9\beta c'^2 \pmod{10}$ | 8 | 4 | 1 | 7 |
| $\mathrm{Arg}_1(d_1 \to d_4;1)$ | $-\frac{2}{10}+\frac{1}{2}$ | $\frac{4}{10}-\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_4;1)$: $\frac{3\beta}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4;1)$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ |
| $3\beta c' \pmod{10}$ | 8 | 8 | 3 | 3 |
| $-18\beta c'^2 \pmod 5$ | 1 | 3 | 2 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_4;2)$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_4;2)$: $\frac{3\beta}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4;2)$ | $-\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ |

Table 8.3: Table for $\mathrm{Arg}(d_1 \to d_4;\ell)$; $2 \nmid c$, $3|c$, $5 \nmid c$.

Next we investigate $\mathrm{Arg}(d_1 \to d_2;\ell)$. For $\mathrm{Arg}_1(d_1 \to d_2;\ell)$, we use (8.16):

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_3}{5})/\sin(\tfrac{\pi a_1}{5})\right) = 1 \quad \text{and} \quad \mathrm{sgn}\left(\sin(\tfrac{2\pi a_3}{5})/\sin(\tfrac{2\pi a_1}{5})\right) = -1$$

By (8.9), we have

$$12cs(d_2,c) - 12cs(d_1,c) \equiv d_2 + \overline{d_{2\{3c\}}} - d_1 - \overline{d_{1\{3c\}}} \equiv \beta c'(1 - \overline{d_{2\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{8.22}$$

As $15|3c$, after dividing by $c'$ we have

$$60s(d_2,c) - 60s(d_1,c) \equiv \beta(1 - \overline{d_{2\{15\}}} \cdot \overline{d_{1\{15\}}}) \equiv \beta(1 - a_3a_1) \pmod{15}. \tag{8.23}$$

Since $a_3 \equiv a_1 \pmod 3$, we have $a_3a_1 \equiv 1 \pmod 3$. We also have $a_3a_1 \equiv 3 \pmod 5$ by (8.3), then $a_3a_1 \equiv 13 \pmod{15}$ and

$$-(60s(d_2,c) - 60s(d_1,c)) \equiv -3\beta \pmod{15}. \tag{8.24}$$

By (8.10) we have

$$12cs(d_4,c) - 12cs(d_1,c) \equiv 4 \pmod 8. \tag{8.25}$$

The congruences (8.24) and (8.25) determines a unique value modulo 120.

| $c'$ (mod 30) | 3 | 9 | 21 | 27 |
|---|---|---|---|---|
| $\beta$ | 2 | 4 | 1 | 3 |
| $-3c'$ (mod 5) | 1 | 3 | 2 | 4 |
| $\text{Arg}_1(d_1 \to d_2; 1)$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ |
| $\text{Arg}_2(d_1 \to d_2; 1)$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ |
| $\text{Arg}_3(d_1 \to d_2; 1)$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ |
| Total $\text{Arg}(d_1 \to d_2; 1)$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $\frac{1}{2}$ |
| $-12c'$ (mod 5) | 4 | 2 | 3 | 1 |
| $\text{Arg}_1(d_1 \to d_2; 2)$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ |
| $\text{Arg}_2(d_1 \to d_2; 2)$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ |
| $\text{Arg}_3(d_1 \to d_2; 2)$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ |
| Total $\text{Arg}(d_1 \to d_2; 2)$ | $-\frac{2}{5}$ | 0 | 0 | $\frac{2}{5}$ |

Table 8.4: Table for $\text{Arg}(d_1 \to d_2; \ell)$; $2 \nmid c$, $3|c$, $5 \nmid c$.

Combining Table 8.3 and Table 8.4 we finish the proof in the case $2 \nmid c'$, $3|c'$ and $5 \nmid c'$.

### 8.1.3 $2|c'$, $3 \nmid c'$, and $5 \nmid c'$

These are the case for $c' \equiv 2, 4, 8, 14, 16, 22, 26, 28 \pmod{30}$. For $\text{Arg}_1(d_1 \to d_4; \ell)$ we still use (8.13):

$$\text{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5})/\sin(\tfrac{\pi a_1 \ell}{5})\right) = -1 \quad \text{whenever} \begin{cases} \ell = 1 \\ 3\beta c' \equiv 8 \pmod{10} \end{cases} \text{or } \ell = 2.$$

By (8.9), $\theta = 1$ and we still have

$$-\left(12cs(d_4, c) - 12cs(d_1, c)\right) \equiv -(d_4 + a_4 - d_1 - a_1) \equiv -6\beta c' \equiv -\beta c' \pmod{c}, \tag{8.26}$$

and $12cs(d, c) \equiv 0 \pmod 6$. Define the integer $\lambda \geq 1$ by $2^\lambda \| c$. To determine the value modulo $24c$, we need to determine it modulo $8 \times 2^\lambda$. By (8.11) we have

$$\begin{aligned} 12cs(d_4, c) - 12cs(d_1, c) &\equiv d_4 - d_1 + (c^2 + 3c + 1)(\overline{d_{4\{8\times 2^\lambda\}}} - \overline{d_{1\{8\times 2^\lambda\}}}) \\ &\quad + 2c\left(\overline{d_{4\{8\times 2^\lambda\}}}(\tfrac{c}{d_4}) - \overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})\right) \pmod{8 \times 2^\lambda} \\ &\equiv 3\beta c'\left(1 - (c^2 + 3c + 1)\overline{d_{4\{8\times 2^\lambda\}}} \cdot \overline{d_{1\{8\times 2^\lambda\}}}\right) \\ &\quad + 2c\left(\overline{d_{4\{8\times 2^\lambda\}}}(\tfrac{c}{d_4}) - \overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})\right) \pmod{8 \times 2^\lambda}. \end{aligned} \tag{8.27}$$

We claim that

$$12cs(d_4, c) - 12cs(d_1, c) \equiv 0 \pmod{8 \times 2^\lambda}. \tag{8.28}$$

To do this, since $2^\lambda \| c'$, $\overline{x_{\{8\}}} \equiv x \pmod 8$ for odd $x$, and $c'|(12cs(d_4,c)-12cs(d_1,c))$ by (8.26), we divide $c'$ in (8.27) and obtain

$$\begin{aligned}60\left(s(d_4,c)-s(d_1,c)\right) &\equiv 3\beta\left(1-(c^2+3c+1)d_4d_1\right)+2\left(d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1})\right)\\ &\equiv 3\beta c'\left(3\beta d_1-1\right)(c'-1)+2\left(d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1})\right) \pmod 8.\end{aligned}$$

Define val.:= $3\beta c'\left(3\beta d_1-1\right)(c'-1) \pmod 8$. Note that both $d_1$ and $c'-1$ are odd. We have Table 8.5 for val.:

| $c' \pmod 5$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 4 |
| $3\beta c'$ | $3c'$ | $6c'$ | $9c'$ | $12c'$ |
| $3\beta d_1-1 \pmod 2$ | $3d_1-1$ | $6d_1-1$ | $9d_1-1$ | $12d_1-1$ |
| $2\|c$, $d_1\equiv 1 \pmod 4$; | 4 | 4 | 0 | 0 |
| $2\|c$, $d_1\equiv 3 \pmod 4$; | 0 | 4 | 4 | 0 |
| $4\|c$; | 0 | 0 | 0 | 0 |

Table 8.5: Table of val.:= $3\beta c'\left(3\beta d_1-1\right)(c'-1) \pmod 8$; $2|c$, no requirement for $(c,3)$, $5\nmid c$.

For the second term we only need to determine $d_4(\frac{c}{d_4})-d_1(\frac{c}{d_1}) \pmod 4$. When $\lambda$ is even, we have $d_4\equiv d_1 \pmod 4$ and $(\frac{2^\lambda}{d_4})=(\frac{2^\lambda}{d_1})=1$; when $\lambda\geq 3$ is odd, we have $d_4\equiv d_1 \pmod 8$ and $(\frac{2}{d_4})=(\frac{2}{d_1})=1$. By quadratic reciprocity,

$$\begin{aligned}d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1}) &\equiv d_1\left((\tfrac{5}{d_4})\left(\tfrac{c'/2^\lambda}{d_4}\right)-(\tfrac{5}{d_4})\left(\tfrac{c'/2^\lambda}{d_1}\right)\right)\\ &\equiv d_1\left(\tfrac{d_1}{c'/2^\lambda}\right)\left((-1)^{(d_4-1)(\frac{c'}{2^\lambda}-1)/4}-(-1)^{(d_1-1)(\frac{c'}{2^\lambda}-1)/4}\right)\equiv 0 \pmod 4\end{aligned}$$

where the last equality is because $\frac{d_4-1}{2}$ and $\frac{d_1-1}{2}$ are of the same parity. This matches the last row (case $4|c$) in Table 8.5.

When $2\|c$, recall $d_4=d_1+3\beta c'$ and we have

$$d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1})\equiv(\tfrac{d_1}{c'/2})\left((\tfrac{2}{d_4})(-1)^{(d_4-1)(\frac{c'}{2}-1)/4}d_4-(\tfrac{2}{d_1})(-1)^{(d_1-1)(\frac{c'}{2}-1)/4}d_1\right) \pmod 4 \tag{8.29}$$

When $c'\equiv 2 \pmod 8$, $\frac{c'/2-1}{2}$ is even and (8.29) gets to $(\frac{2}{d_4})d_4-(\frac{2}{d_1})d_1 \pmod 4$; When $c'\equiv 6 \pmod 8$, $\frac{c'/2-1}{2}$ is odd and (8.29) gets to $(\frac{2}{d_4})(-1)^{\frac{d_4-1}{2}}d_4-(\frac{2}{d_1})(-1)^{\frac{d_1-1}{2}}d_1 \pmod 4$. Since $c=5c'\equiv c' \pmod 8$, we can use $d_4=d_1+3\beta c'$ to determine $d_4 \pmod 8$ and get Table 8.6.

| (8.29) $\searrow$ | $c'\equiv 2 \pmod 8$ | | | | $c'\equiv 6 \pmod 8$ | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1 \pmod 8$ | 1 | 3 | 5 | 7 | 1 | 3 | 5 | 7 |
| $\beta=1$, (8.29) | 2 | 0 | 2 | 0 | 2 | 0 | 2 | 0 |
| $\beta=2$, (8.29) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\beta=3$, (8.29) | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 2 |
| $\beta=4$, (8.29) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 8.6: Table for (8.29); $2|c$, no requirement for $(c,3)$, $5\nmid c$.

Comparing Table 8.5 and Table 8.6 proves (8.28). Recall (8.8) and (8.26), we divide both the denominator and numerator in $P_2$ by $24c'$ and get $\mathrm{Arg}_2(d_1\to d_4;\ell)=e(\frac{\beta}{5})$. Since $\mathrm{Arg}_3(d_1\to d_4;\ell)=e(\frac{2\beta}{5})$, we have

Table 8.7.

| $c'$ (mod 30) | 2 | 4 | 8 | 14 | 16 | 22 | 26 | 28 |
|---|---|---|---|---|---|---|---|---|
| $\beta$ | 3 | 4 | 2 | 4 | 1 | 3 | 1 | 2 |
| $3\beta c'$ (mod 10) | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 |
| $-9\beta c'^2$ (mod 10) | 2 | 4 | 8 | 4 | 6 | 2 | 6 | 8 |
| $\mathrm{Arg}_1(d_1 \to d_4; 1)$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 1)$: $\frac{3\beta}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4; 1)$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ |
| $-18\beta c'^2 \equiv 2c'$ (mod 5) | 4 | 3 | 1 | 3 | 2 | 4 | 2 | 1 |
| $\mathrm{Arg}_1(d_1 \to d_4; 2)$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 2)$: $\frac{3\beta}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4; 2)$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $-\frac{1}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $\frac{1}{2}$ | $-\frac{1}{10}$ |

Table 8.7: Table for $\mathrm{Arg}(d_1 \to d_4; 2)$; $2|c$, $3 \nmid c$, $5 \nmid c$.

Next we deal with $\mathrm{Arg}(d_1 \to d_2; \ell)$. For $\mathrm{Arg}_1(d_1 \to d_2; \ell)$, we still use (8.16):

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_3}{5})/\sin(\tfrac{\pi a_1}{5})\right) = 1 \quad \text{and} \quad \mathrm{sgn}\left(\sin(\tfrac{2\pi a_3}{5})/\sin(\tfrac{2\pi a_1}{5})\right) = -1$$

By (8.9),

$$-\left(12cs(d_2, c) - 12cs(d_1, c)\right) \equiv -(d_2 + a_3 - d_1 - a_1) \equiv -3\beta c' \equiv 2\beta c' \pmod{c}. \tag{8.30}$$

This congruence shows that $12cs(d_2, c) - 12cs(d_1, c)$ is divisible by $c'$. Denote $\lambda$ by $2^\lambda \| c$. We claim that

$$-\left(12cs(d_2, c) - 12cs(d_1, c)\right) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}. \tag{8.31}$$

To prove (8.31), we apply (8.11) to get

$$\begin{aligned} 12cs(d_2, c) - 12cs(d_1, c) \equiv\ & \beta c' \left(1 - (c^2 + 3c + 1)\overline{d_{4\{8\times 2^\lambda\}}} \cdot \overline{d_{1\{8\times 2^\lambda\}}}\right) \\ & + 2c\left(\overline{d_{2\{8\times 2^\lambda\}}}(\tfrac{c}{d_2}) - \overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})\right) \pmod{8 \times 2^\lambda}. \end{aligned} \tag{8.32}$$

Then similar as (8.27),

$$60s(d_2, c) - 60s(d_1, c) \equiv \beta c'(\beta d_1 - 1)(c' - 1) + 2\left(d_2(\tfrac{c}{d_2}) - d_1(\tfrac{c}{d_1})\right) \pmod{8}. \tag{8.33}$$

See Table 8.8 for the first part val.:$= \beta c'(\beta d_1 - 1)(c' - 1)$ and note that $d_1$ is odd and $c' - 1$ is odd:

| $c'$ (mod 5) | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 4 |
| $\beta c'$ | $c'$ | $3c'$ | $2c'$ | $4c'$ |
| $\beta d_1 - 1$ | $d_1 - 1$ | $3d_1 - 1$ | $2d_1 - 1$ | $4d_1 - 1$ |
| $2\|c$, $d_1 \equiv 1$ (mod 4); val. (mod 8): | 0 | 4 | 4 | 0 |
| $2\|c$, $d_1 \equiv 3$ (mod 4); val. (mod 8): | 4 | 0 | 4 | 0 |
| $4|c$; val. (mod 8): | 0 | 0 | 0 | 0 |

Table 8.8: Table for val.:$= \beta c'(\beta d_1 - 1)(c' - 1)$ (mod 8); $2|c$, no requirement for $(c, 3)$, $5 \nmid c$.

For the second part $2\left(d_2(\frac{c}{d_2}) - d_1(\frac{c}{d_1})\right)$ (mod 8), we do a similar process as Table 8.6 by quadratic reciprocity and skip this step. Combining (8.8), (8.30) and (8.31), we have

$$-(12cs(d_2,c) - 12cs(d_1,c)) \equiv 12 \times 2^{\lambda} \pmod{24 \times 2^{\lambda}}.$$

By dividing $c'$, $-60s(d_2,c) + 60s(d_1,c) \pmod{120}$ is uniquely determined by $2\beta \pmod 5$ and $12 \pmod{24}$. Hence

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{1,7,3,9}{10}, \quad \text{for } \beta = 1,2,3,4, \text{ respectively}$$

and we get Table 8.9.

| $c' \pmod{30}$ | 2 | 4 | 8 | 14 | 16 | 22 | 26 | 28 |
|---|---|---|---|---|---|---|---|---|
| $\beta$ | 3 | 4 | 2 | 4 | 1 | 3 | 1 | 2 |
| $2\beta c' \pmod{10}$ | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $-3\beta c'^2 \equiv 2c' \pmod 5$ | 4 | 3 | 1 | 3 | 2 | 4 | 2 | 1 |
| $\mathrm{Arg}_1(d_1 \to d_2; 1)$ | $\frac{4}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{1}{5}$ |
| $\mathrm{Arg}_2(d_1 \to d_2; 1)$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ |
| $\mathrm{Arg}_3(d_1 \to d_2; 1)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{3}{10}$ | $\frac{1}{2}$ |
| $-12\beta c'^2 \equiv 3c' \pmod 5$ | 1 | 2 | 4 | 2 | 3 | 1 | 3 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2)$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ |
| $\mathrm{Arg}_2(d_1 \to d_2; 2)$ | $\frac{3}{10}$ | $-\frac{1}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $\frac{1}{10}$ | $-\frac{3}{10}$ |
| $\mathrm{Arg}_3(d_1 \to d_2; 2)$ | $\frac{2}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{3}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $\frac{2}{5}$ | 0 | $-\frac{2}{5}$ | 0 | 0 | $\frac{2}{5}$ | 0 | $-\frac{2}{5}$ |

Table 8.9: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3 \nmid c$, $5 \nmid c$.

Comparing Table 8.7 and Table 8.9, we confirm that Condition 8.1 is satisfied in these cases.

### 8.1.4 $2|c'$, $3|c'$, and $5 \nmid c'$

These are the cases $c' \equiv 6, 12, 18, 24 \pmod{30}$. For $\mathrm{Arg}_1(d_1 \to d_4; \ell)$ we use (8.13):

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5})/\sin(\tfrac{\pi a_1 \ell}{5})\right) = -1 \quad \text{whenever} \begin{cases} \ell = 1 \\ 3\beta c' \equiv 8 \pmod{10} \end{cases} \quad \text{or } \ell = 2.$$

For $\mathrm{Arg}_2(d_1 \to d_4; \ell)$, by (8.9) we have

$$-(12cs(d_4,c) - 12cs(d_1,c)) \equiv -3\beta c'(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{8.34}$$

The proof of (8.28) in the former subsection still works for $3|c$. Then $-(12cs(d_4,c) - 12cs(d_1,c))$ is a multiple of $24c'$. After dividing both the denominator and numerator in $P_2$ and recalling $\overline{d_{j\{3c\}}} \equiv a_j \equiv j \pmod 5$ for $j = 1,4$, we get $\mathrm{Arg}_2(d_1 \to d_4; \ell) = e(\frac{\beta}{5})$. Now we have Table 8.10.

Then we check $\mathrm{Arg}(d_1 \to d_2; \ell)$. For $\mathrm{Arg}_1(d_1 \to d_2; \ell)$, we use (8.16):

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_3}{5})/\sin(\tfrac{\pi a_1}{5})\right) = 1 \quad \text{and} \quad \mathrm{sgn}\left(\sin(\tfrac{2\pi a_3}{5})/\sin(\tfrac{2\pi a_1}{5})\right) = -1.$$

| $c'$ (mod 30) | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 4 |
| $3\beta c'$ (mod 10) | 8 | 8 | 8 | 8 |
| $-9\beta c'^2$ (mod 10) | 6 | 2 | 8 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_4; 1)$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 1) : \frac{3\beta}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4; 1)$ | $-\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{10}$ |
| $-18\beta c'^2 \equiv 2c'$ (mod 5) | 2 | 4 | 1 | 3 |
| $\mathrm{Arg}_1(d_1 \to d_4; 2)$ | $-\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $\frac{1}{10}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 2) : \frac{3\beta}{5}$ | $\frac{3}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_4; 2)$ | $\frac{1}{2}$ | $\frac{1}{10}$ | $-\frac{1}{10}$ | $\frac{1}{2}$ |

Table 8.10: Table for $\mathrm{Arg}(d_1 \to d_4; \ell)$; $2|c$, $3|c$, $5 \nmid c$.

For $\mathrm{Arg}_2(d_1 \to d_2; \ell)$, by (8.9) we have

$$-\left(12cs(d_2, c) - 12cs(d_1, c)\right) \equiv -\beta c'(1 - \overline{d_{2\{3c\}} d_{1\{3c\}}}) \pmod{3c}. \tag{8.35}$$

Since $3|c$, $\overline{d_{2\{3c\}}} \equiv a_3 \pmod{15}$ and $\overline{d_{1\{3c\}}} \equiv a_1 \pmod{15}$. After dividing by $c'$ we have

$$-(60s(d_2, c) - 60s(d_1, c)) \equiv -\beta(1 - a_3 a_1) \pmod{15}.$$

Combining $a_3 = a_1 + 2\beta c'$ and $a_3 a_1 \equiv 3 \pmod 5$ we get $a_3 a_1 \equiv 13 \pmod{15}$ and

$$-\left(60s(d_2, c) - 60s(d_1, c)\right) \equiv -3\beta \pmod{15}. \tag{8.36}$$

Denote $\lambda$ by $2^\lambda \| c$, then (8.31) still works as

$$-\left(60s(d_2, c) - 60s(d_1, c)\right) \equiv 4 \pmod 8. \tag{8.37}$$

By(8.36) and (8.37),

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{1, 7, 3, 9}{10} \quad \text{for } \beta = 1, 2, 3, 4, \text{ respectively.}$$

This gives Table 8.11.

Comparing Table 8.10 and Table 8.11, we have proved that Condition 8.1 is satisfied in these cases.

### 8.1.5 $5|c'$

The case when $25|c$ and is different from the former cases. We still denote $c' = c/5$ and $V(r, c) := \{d \pmod{c}^* : d \equiv r \pmod{c'}\}$ for $r \pmod{c'}^*$. Now $|V(r, c)| = 5$ and since $(d + c', c) = 1$ when $(d, c) = 1$, we can write $V(r, c) = \{d, d + c', d + 2c', d + 3c', d + 4c'\}$ for $1 \le d < c'$ and $d \equiv r \pmod{c'}$.

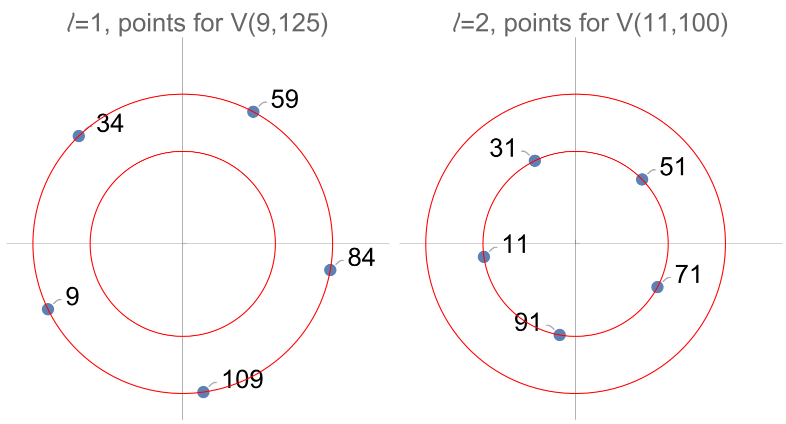
We claim that (8.1) is still true:

$$\sum_{d \in V(r,c)} \frac{e\left(-\frac{3c' a \ell^2}{10}\right)}{\sin(\frac{\pi a \ell}{p})} e\left(-\frac{12cs(d, c)}{24c}\right) e\left(\frac{4d}{c}\right) = 0, \tag{8.38}$$

| $c'$ (mod 30) | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 4 |
| $-3\beta c'^2 \equiv 2c'$ (mod 5) | 2 | 4 | 1 | 3 |
| $\mathrm{Arg}_1(d_1 \to d_2; 1)$ | $\frac{2}{5}$ | $\frac{4}{5}$ | $\frac{1}{5}$ | $\frac{3}{5}$ |
| $\mathrm{Arg}_2(d_1 \to d_2; 1)$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_3(d_1 \to d_2; 1)$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $\frac{3}{10}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $-\frac{3}{10}$ |
| $-12\beta c'^2 \equiv 3c'$ (mod 5) | 3 | 1 | 4 | 2 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2)$ | $\frac{1}{10}$ | $-\frac{3}{10}$ | $\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_2(d_1 \to d_2; 2)$ | $\frac{1}{10}$ | $\frac{3}{10}$ | $-\frac{3}{10}$ | $-\frac{1}{10}$ |
| $\mathrm{Arg}_3(d_1 \to d_2; 2)$ | $\frac{4}{5}$ | $\frac{2}{5}$ | $\frac{3}{5}$ | $\frac{1}{5}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | 0 | $\frac{2}{5}$ | $-\frac{2}{5}$ | 0 |

Table 8.11: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3|c$, $5 \nmid c$.

but this time we have five summands. We prove (8.38) by showing that there are only two cases for the sum:




i.e. all at the outer circle (radius $1/\sin(\frac{\pi}{5})$) or all at the inner circle (radius $1/\sin(\frac{2\pi}{5})$) and equally distributed. As before, we still denote $P_1$, $P_2$ and $P_3$ for each term in (8.38) and investigate the argument differences contributed from each term. Note that $P_1(d) = (-1)^{ca\ell}/\sin(\frac{\pi a \ell}{5})$ has period $c'$, hence $\mathrm{Arg}_1(d \to d_1; \ell) = 0$ always.

For any $d \in V(r, c)$, we take $a$ (mod $c$) such that $ad \equiv 1$ (mod $c$). We denote $d_* = d + c'$ and $a_*$ by $a_* d_* \equiv 1$ (mod $c$). Then can pick $a_* = a - c'$ when $d \equiv 1, 4$ (mod 5) and pick $a_* = a + c'$ when $d \equiv 2, 3$ (mod 5).

In the following two cases, we prove $\mathrm{Arg}(d \to d_*; 1)$ is constant and independent from the choice of $d \in V(r, c)$. The other case $\ell = 2$ only affects $P_1$ (radii for those five points) and we still get (8.38).

#### 8.1.5.1 $c$ is odd

When $d \equiv 1, 4$ (mod 5) and $3 \nmid c$, (8.8), (8.9) and (8.10) imply that

$$12cs(d_*, c) - 12cs(d, c) \equiv 0 \pmod{24c}, \tag{8.39}$$

hence $\mathrm{Arg}_2(d \to d_*; \ell) = 0$ always. As $\mathrm{Arg}_3(d \to d_*; \ell) = \frac{4}{5}$ for any $d \in V(r, c)$, we have proved (8.38) in this case.

When $3|c$ and $d \equiv 1, 4 \pmod 5$, (8.9) implies

$$-(12cs(d_*, c) - 12cs(d, c)) \equiv -c'(1 - \overline{d_{*\{3c\}}} \cdot \overline{d_{\{3c\}}}) \pmod{3c}. \tag{8.40}$$

Since $15|c$, after dividing by $c'$ we have

$$-(60s(d_*, c) - 60s(d, c)) \equiv a^2 - 1 \pmod{15}. \tag{8.41}$$

Note that $a \equiv 1, 4 \pmod 5$ and $a^2 \equiv 1 \pmod{15}$, hence we have $-(12cs(d_*, c) - 12cs(d, c)) \equiv 0 \pmod{24c}$ and the same conclusion as the former case.

When $d \equiv 2, 3 \pmod 5$ and $3 \nmid c$, recall $d_* = d + c'$ and $a_1 = a + c'$ with $a + d \equiv 0 \pmod 5$. By (8.8), (8.9) and (8.10), we have

$$-(12cs(d_*, c) - 12cs(d, c)) \equiv -2c' \pmod c \quad \text{and} \quad \equiv 0 \pmod{24}.$$

Then $\mathrm{Arg}_2(d \to d_*; \ell) = \frac{2}{5}$. Since $\mathrm{Arg}_3(d \to d_*; \ell) = \frac{4}{5}$, we have proved this case.

When $d \equiv 2, 3 \pmod 5$ and $3|c$, we still get (8.41), while this time $a \equiv 3, 2 \pmod 5$, $a^2 - 1 \equiv 3 \pmod{15}$, and hence $a^2 - 1 \equiv 48 \pmod{120}$. We have $\mathrm{Arg}_2(d \to d_*; \ell) = \frac{2}{5}$. Since $\mathrm{Arg}_3(d \to d_*; \ell) = \frac{4}{5}$, we have proved this case.

#### 8.1.5.2 $c$ is even

In this case, denote $\lambda$ by $2^\lambda \| c$. Then by (8.11) we have

$$\begin{aligned} 12cs(d_*, c) - 12cs(d, c) \equiv c' &\left(1 - (c^2 + 3c + 1)\overline{d_{1\{8\times 2^\lambda\}}} \cdot \overline{d_{\{8\times 2^\lambda\}}}\right) \\ &+ 2c\left((\tfrac{c}{d_*})\overline{d_{1\{8\times 2^\lambda\}}} - (\tfrac{c}{d})\overline{d_{\{8\times 2^\lambda\}}}\right) \pmod{8 \times 2^\lambda}. \end{aligned}$$

Since $c' | (12cs(d_*, c) - 12cs(d, c))$ by (8.39) and (8.40), dividing the above congruence by $c'$ we have

$$-60(s(d_*, c) - s(d, c)) \equiv -c'(d-1)(c'-1) - 2\left((\tfrac{c}{d_*})d_* - (\tfrac{c}{d})d\right) \pmod 8. \tag{8.42}$$

For the first term,

$$-c'(d-1)(c'-1) \equiv \begin{cases} 0 \pmod 8 & \text{if } 2\|c,\ d \equiv 1 \pmod 4; \\ 4 \pmod 8 & \text{if } 2\|c,\ d \equiv 3 \pmod 4; \\ 0 \pmod 8 & \text{if } 4|c. \end{cases} \tag{8.43}$$

When $\lambda$ is even, $(\frac{2^\lambda}{d_*}) = (\frac{2^\lambda}{d}) = 1$; when $\lambda \geq 3$ is odd, $(\frac{2}{d_*}) = (\frac{2}{d})$. In either case $\frac{d_*-1}{2}$ and $\frac{d-1}{2}$ have the same parity. Hence when $4|c$, we have

$$(\tfrac{c}{d_*})d_* - (\tfrac{c}{d})d \equiv 0 \pmod 4.$$

When $2\|c$, we have Table 8.12 for val.:= $(\frac{c}{d_*})d_* - (\frac{c}{d})d \pmod 4$ using quadratic reciprocity.

Combining (8.43) and Table 8.12, for $2^\lambda \| c$ we get

$$12cs(d_*, c) - 12cs(d, c) \equiv 0 \pmod{8 \times 2^\lambda}. \tag{8.44}$$

The argument for the cases $d \equiv 1, 4 \pmod 5$ or $d \equiv 2, 3 \pmod 5$, or the cases $3 \nmid c$ or $3|c$, are the same as the former case $c$ odd and we hold the same conclusion.

| $d \pmod 8$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $d_* \pmod 8$ when $c' \equiv 2 \pmod 8$ | 3 | 5 | 7 | 1 |
| val. | 0 | 2 | 0 | 2 |
| $d_* \pmod 8$ when $c' \equiv 6 \pmod 8$ | 7 | 1 | 3 | 5 |
| val. | 0 | 2 | 0 | 2 |

Table 8.12: Table for val.:= $(\frac{c}{d_*})d_* - (\frac{c}{d})d \pmod 4$; $2|c$, no requirement for $(3,c)$, $5|c$.

This finishes the proof of claim (1) in Theorem 1.15.

## 8.2 Proof of Theorem 1.15, claim (2)

Recall Proposition 7.1:

$$S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7) = e(-\tfrac{1}{8}) \sum_{d \pmod c^*} \frac{\overline{\mu(c,d,[a\ell],7)}}{\sin(\frac{\pi[a\ell]}{7})} e^{-\pi i s(d,c)} e\left(\frac{(7n+5)d}{c}\right)$$

where $ad \equiv 1 \pmod c$. We only need to consider $\ell = 1,2,3$ because $A(\frac{\ell}{p};n) = A(1-\frac{\ell}{p};n)$.

Let $c = 7c'$. For $r \pmod{c'}^*$, we still define

$$V(r,c) = \{d \pmod c^* : \ d \equiv r \pmod{c'}\}.$$

For example, $V(1,42) = \{1,13,19,25,31,37\}$ and $V(4,35) = \{4,9,19,24,29,34\}$. It is not hard to show that $|V(r,c)| = 6$ if $7 \nmid c'$, $|V(r,c)| = 7$ if $49|c$, and $(\mathbb{Z}/c\mathbb{Z})^*$ is the disjoint union

$$(\mathbb{Z}/c\mathbb{Z})^* = \bigcup_{r \pmod{c'}^*} V(r,c), \quad \text{where } V(r_1,c) \cap V(r_2,c) = \emptyset \text{ for } r_1 \not\equiv r_2 \pmod{c'}.$$

As in (7.23), we have

$$\frac{\overline{\mu(c,d,[a\ell],7)}}{\sin(\frac{\pi[a\ell]}{7})} = \frac{(-1)^{\ell c}}{\sin(\frac{\pi a\ell}{7})} \exp\left(-\frac{3\pi i c' a\ell^2}{7}\right).$$

We claim that for $\ell = 1,2,3$, the sum on $V(r,c)$ for all $r \pmod{c'}^*$ is zero:

$$\sum_{d \in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})} e\left(-\frac{12cs(d,c)}{24c}\right) e\left(\frac{5d}{c}\right) = 0 \tag{8.45}$$

If this is true, then

$$\sum_{d \in V(r,c)} \frac{\overline{\mu(c,d,[a\ell],7)}}{\sin(\frac{\pi[a\ell]}{7})} e^{-\pi i s(d,c)} e\left(\frac{(7n+5)d}{c}\right) = e\left(\frac{nr}{c'}\right)(-1)^{\ell c} \cdot 0 = 0$$

for all $n \in \mathbb{Z}$, $\ell = 1,2,3$ and we have proved claim (2) of Theorem 1.15.

We define each term in (8.45) as

$$P(d) := \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})} \cdot e\left(-\frac{12cs(d,c)}{24c}\right) \cdot e\left(\frac{5d}{c}\right) =: P(d_1) \cdot P(d_2) \cdot P(d_3).$$

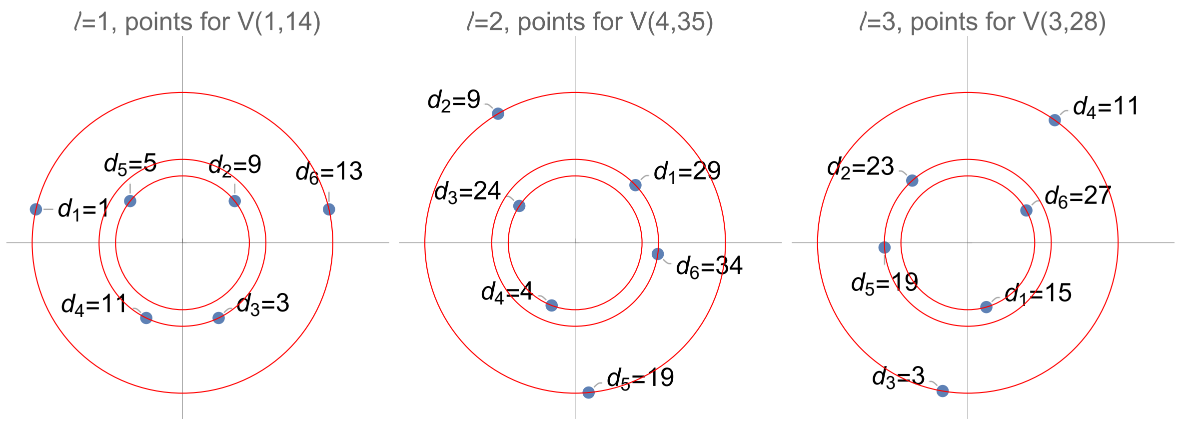
First, we deal with the case $7 \nmid c'$. We still denote as (8.5) for the argument differences, but in this case $u, v \in \{1, 2, \cdots, 6\}$ and $\ell \in \{1, 2, 3\}$, where

$$d_u \equiv a_u \equiv u \pmod 7, \;\; a_{\overline{u_{\{7\}}}} d_u \equiv 1 \pmod c, \;\; d_{u+1} = d_u + \beta c' \text{ and } a_{u+1} = a_u + \beta c'. \tag{8.46}$$

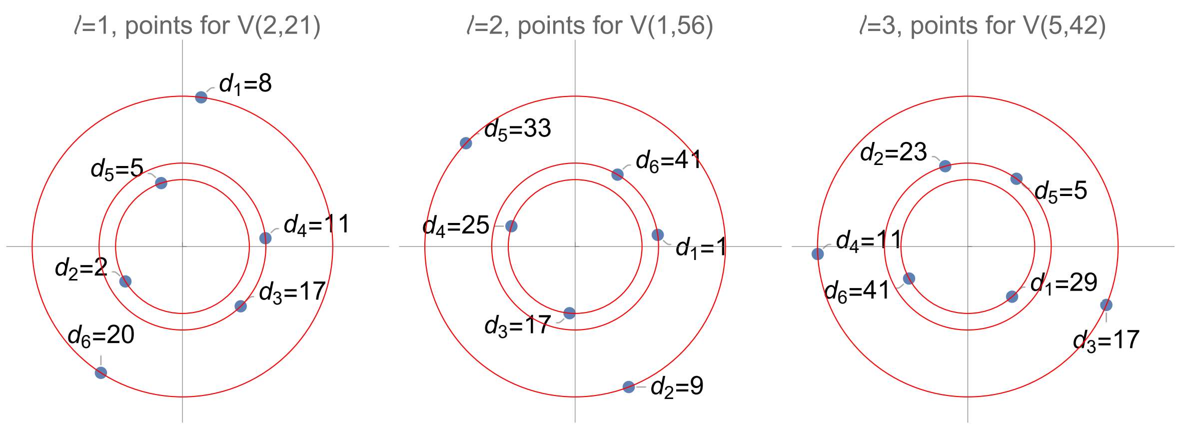
Note that the second congruence is not for $a_u d_u$ but due to $\overline{u_{\{7\}}} \cdot u \equiv 1 \pmod 7$. Let $1 \le \beta \le 6$ such that $\beta c' \equiv 1 \pmod 7$.

As in Condition 8.1, we have the following styles for the six summands followed by the explanation in Condition 8.3:

- $\ell = 1, 2, 3$, first style.




- $\ell = 1, 2, 3$, reversed style from the above.




Here we explain these styles. Each graph above includes three circles centered at the origin with radii $\csc(\frac{\pi}{7})$, $\csc(\frac{2\pi}{7})$ and $\csc(\frac{3\pi}{7})$, respectively. The six points in each graph above mark $P(d)$ for $d \in V(r, c)$ on these three circles. It is not hard to prove that whenever the six points satisfy the following condition on their argument differences, they sum to zero. This proves (8.45). One hint is the equation

$$\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})} - \frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})} + \frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})} = 0, \quad \text{where } \frac{1}{\sin(\frac{\pi}{7})}, \; \frac{1}{\sin(\frac{2\pi}{7})}, \; \frac{1}{\sin(\frac{3\pi}{7})} \text{ are the radii.}$$

**Condition 8.3.** *We have the following six styles for these six points.*

- $\ell = 1$*: the arguments (as a proportion of* $2\pi$*) going* $d_1 \to d_2 \to d_3 \to d_4 \to d_5 \to d_6 \to d_1$ *are* $-\frac{5}{14}$, $-\frac{2}{7}$, $-\frac{1}{7}$, $-\frac{2}{7}$, $-\frac{5}{14}$, *and* $\frac{3}{7}$, *respectively, or the reversed style.*

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 2, 4 \pmod 7$ | | $-\frac{5}{14}$ | | $-\frac{2}{7}$ | | $-\frac{1}{7}$ | | $-\frac{2}{7}$ | | $-\frac{5}{14}$ | | $\frac{3}{7}$ | |
| $c' \equiv 3, 5 \pmod 7$ | | $\frac{5}{14}$ | | $\frac{2}{7}$ | | $\frac{1}{7}$ | | $\frac{2}{7}$ | | $\frac{5}{14}$ | | $-\frac{3}{7}$ | |

- $\ell = 2$, *second graph style:*

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 5, 6 \pmod 7$ | | $\frac{3}{14}$ | | $\frac{1}{14}$ | | $\frac{2}{7}$ | | $\frac{1}{14}$ | | $\frac{3}{14}$ | | $\frac{1}{7}$ | |
| $c' \equiv 1, 2 \pmod 7$ | | $-\frac{3}{14}$ | | $-\frac{1}{14}$ | | $-\frac{2}{7}$ | | $-\frac{1}{14}$ | | $-\frac{3}{14}$ | | $-\frac{1}{7}$ | |

- $\ell = 3$, *third graph style:*

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 1, 4 \pmod 7$ | | $-\frac{3}{7}$ | | $\frac{5}{14}$ | | $\frac{3}{7}$ | | $\frac{5}{14}$ | | $-\frac{3}{7}$ | | $-\frac{2}{7}$ | |
| $c' \equiv 3, 6 \pmod 7$ | | $\frac{3}{7}$ | | $-\frac{5}{14}$ | | $-\frac{3}{7}$ | | $-\frac{5}{14}$ | | $\frac{3}{7}$ | | $\frac{2}{7}$ | |

*Remark.* Note that claim (2) of Theorem 1.15 is for the case $c'\ell \not\equiv \pm 1 \pmod 7$, so Condition 8.3 does not include all the cases of $c' \pmod 7$. We will highlight these exceptional cases among the tables in this section by a row "$c'\ell \equiv \pm 1 \pmod 7$?". The corresponding entry is:

$$\begin{cases} \text{blank}, & \text{if } c'\ell \not\equiv \pm 1 \pmod 7; \\ \text{“}+\text{”}, & \text{if } c'\ell \equiv 1 \pmod 7; \\ \text{“}-\text{”}, & \text{if } c'\ell \equiv -1 \pmod 7. \end{cases}$$

We will explain these exceptional styles $c'\ell \equiv \pm 1 \pmod 7$ in the next section for claim (3).

In the following subsections, we show $\operatorname{Arg}(d_1 \to d_2; \ell)$, $\operatorname{Arg}(d_2 \to d_3; \ell)$, and $\operatorname{Arg}(d_3 \to d_4; \ell)$ in all the cases $c' \pmod{42}$. These argument differences are sufficient to check Condition 8.3 because

$$\operatorname{Arg}(d_1 \to d_2; \ell) = \operatorname{Arg}(d_5 \to d_6; \ell) \text{ and } \operatorname{Arg}(d_2 \to d_3; \ell) = \operatorname{Arg}(d_4 \to d_5; \ell),$$

where the proof is the same as the proof of Lemma 8.2.

### 8.2.1 $2 \nmid c', 3 \nmid c', 7 \nmid c'$

We begin by dealing with $\operatorname{Arg}(d_1 \to d_2; \ell)$:

$$\operatorname{Arg}_1(d_1 \to d_2; \ell) = -\frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \operatorname{sgn}(\sin(\frac{\pi a_4 \ell}{7}) / \sin(\frac{\pi a_1 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \operatorname{sgn}(\sin(\frac{\pi a_4 \ell}{7}) / \sin(\frac{\pi a_1 \ell}{7})) = -1. \end{cases}$$

When $\ell = 1$, the sign changes when $3\beta c' \equiv 10 \pmod{14}$. When $\ell = 2$, the sign always changes. When $\ell = 3$, the sign changes when $9\beta c' \equiv 9 \pmod{14}$ but does not change when $9\beta c' \equiv 2 \pmod{14}$.

Since $12cs(d,c) \equiv 0 \pmod 6$, we have

$$-12cs(d_2,c) + 12cs(d_1,c) \equiv -d_2 - a_4 + d_1 + a_1 \equiv -4\beta c' \pmod c,$$

$$-12cs(d_2,c) + 12cs(d_1,c) \equiv 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) - 2(\tfrac{d_1}{7})(\tfrac{d_1}{c'}) \equiv 0 \pmod 8,$$

from which

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{\overline{24_{\{7\}}} \cdot 4\beta}{7} = \frac{\beta}{7}$$

(recall that $\mathrm{Arg}_j(d_u \to d_v; \ell) = x$ means $\mathrm{Arg}_j(d_u \to d_v; \ell) - x \in \mathbb{Z}$). Moreover, $\mathrm{Arg}_3(d_1 \to d_2; \ell) = \frac{5\beta}{7}$. This gives Table 8.13. Note that there are 12 choices of $c'$ so we break the table into upper (for $c' \equiv 1, 5, 11, 13, 17, 19 \pmod 7$) and lower (for $c' \equiv 23, 25, 29, 31, 37, 41 \pmod 7$) parts.

Next we consider $\mathrm{Arg}(d_2 \to d_3; \ell)$, with $d_2 a_4 \equiv d_3 a_5 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_2 \to d_3; \ell) = -\frac{3\beta c'^2 \ell^2}{14} \begin{cases} +0 & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7})/\sin(\frac{\pi a_4 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7})/\sin(\frac{\pi a_4 \ell}{7})) = -1. \end{cases}$$

Note that when $\ell = 1$, the sign changes when $\beta c' \equiv 8 \pmod{14}$. When $\ell = 2$, the sign remains the same. when $\ell = 3$, the sign changes when $3\beta c' \equiv 3 \pmod{14}$ but remains when $10 \pmod{14}$ because $a_4 \ell \equiv 5 \pmod 7$.

Since $12cs(d,c) \equiv 0 \pmod 6$, we have

$$-12cs(d_3,c) + 12cs(d_2,c) \equiv -d_3 - a_5 + d_2 + a_4 \equiv -2\beta c' \pmod c,$$

$$-12cs(d_3,c) + 12cs(d_2,c) \equiv 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) - 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) \equiv 4 \pmod 8,$$

and $-84s(d_3,c) + 84s(d_2,c) \pmod{168}$ is uniquely determined by $12 \pmod{24}$ and $-2\beta \pmod 7$. So

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 1, 3, 5, 2, 4, 6, \text{ resp.}$$

Moreover, $\mathrm{Arg}_3(d_2 \to d_3; \ell) = \frac{5\beta}{7}$. This gives Table 8.14, which is broken into upper (for $c' \equiv 1, 5, 11, 13, 17, 19 \pmod 7$) and lower (for $c' \equiv 23, 25, 29, 31, 37, 41 \pmod 7$) parts.

Then we investigate $\mathrm{Arg}(d_3 \to d_4; \ell)$ with $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$. First we have

$$\mathrm{Arg}_1(d_3 \to d_4; \ell) = \frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7})/\sin(\frac{\pi a_5 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7})/\sin(\frac{\pi a_5 \ell}{7})) = -1. \end{cases}$$

When $\ell = 1$, the sign changes if $3\beta c' \equiv 10 \pmod{14}$. When $\ell = 2$, the sign always changes. When $\ell = 3$, the sign changes if $9\beta c' \equiv 2 \pmod{14}$ but remains if $9\beta c' \equiv 9 \pmod{14}$.

We have $12cs(d,c) \equiv 0 \pmod 6$,

$$-12cs(d_4,c) + 12cs(d_3,c) \equiv 2\beta c' \pmod c,$$

and

$$-12cs(d_4,c) + 12cs(d_3,c) \equiv 2(\tfrac{d_4}{7})(\tfrac{d_4}{c'}) - 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) \equiv 4 \pmod 8.$$

| $c'$ (mod 42) | 1 | 5 | 11 | 13 | 17 | 19 |
|---|---|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $3\beta c'$ (mod 14) | 3 | 3 | 10 | 10 | 3 | 3 |
| $-9\beta c'^2$ (mod 14) | 5 | 11 | 6 | 2 | 1 | 11 |
| $\mathrm{Arg}_1(d_1 \to d_2; 1)$ | $\frac{5}{14}$ | $\frac{11}{14}$ | $\frac{6}{14} + \frac{1}{2}$ | $\frac{2}{14} + \frac{1}{2}$ | $\frac{1}{14}$ | $\frac{11}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{5}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $+$ | | | $-$ | | |
| $-18\beta c'^2 \equiv 3c'$ (mod 7) | 3 | 1 | 5 | 4 | 2 | 1 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $-\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 2)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $-\frac{3}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | | $-$ | |
| $9\beta c'$ (mod 14) | 9 | 9 | 2 | 2 | 9 | 9 |
| $-81\beta c'^2$ (mod 14) | 3 | 1 | 12 | 4 | 9 | 1 |
| $\mathrm{Arg}_1(d_1 \to d_2; 3)$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 3)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 3)$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | | | $+$ |
| $c'$ (mod 42) | 23 | 25 | 29 | 31 | 37 | 41 |
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $3\beta c'$ (mod 14) | 10 | 10 | 3 | 3 | 10 | 10 |
| $-9\beta c'^2$ (mod 14) | 10 | 6 | 5 | 1 | 10 | 2 |
| $\mathrm{Arg}_1(d_1 \to d_2; 1)$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $-\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ |
| $c'\ell = \pm 1$ (mod 7)? | | | $+$ | | | $-$ |
| $-18\beta c'^2 \equiv 3c'$ (mod 7) | 6 | 5 | 3 | 2 | 8 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | $-$ | | |
| $9\beta c'$ (mod 14) | 2 | 2 | 9 | 9 | 2 | 2 |
| $-81\beta c'^2$ (mod 14) | 6 | 12 | 3 | 9 | 6 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_2; 3)$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 3)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | $-$ | |

Table 8.13: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 1 | 5 | 11 | 13 | 17 | 19 |
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $\beta c'$ (mod 14) | 1 | 1 | 8 | 8 | 1 | 1 |
| $-3\beta c'^2$ (mod 14) | 11 | 13 | 2 | 10 | 5 | 13 |
| $\mathrm{Arg}_1(d_2 \to d_3; 1)$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{2}{14}+\frac{1}{2}$ | $\frac{10}{14}+\frac{1}{2}$ | $\frac{5}{14}$ | $\frac{13}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 1)$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $+$ | | | $-$ | | |
| $-6\beta c'^2 \equiv c' \pmod 7$ | 1 | 5 | 4 | 6 | 3 | 5 |
| $\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$ | $\frac{1}{7}$ | $\frac{5}{7}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 2)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | $-$ | |
| $3\beta c'$ (mod 14) | 3 | 3 | 10 | 10 | 3 | 3 |
| $-27\beta c'^2$ (mod 14) | 1 | 5 | 4 | 6 | 3 | 5 |
| $\mathrm{Arg}_1(d_2 \to d_3; 3)$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 3)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 3)$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | | | $+$ |
| $c'$ (mod 42) | 23 | 25 | 29 | 31 | 37 | 41 |
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $\beta c'$ (mod 14) | 8 | 8 | 1 | 1 | 8 | 8 |
| $-3\beta c'^2$ (mod 14) | 8 | 2 | 11 | 5 | 8 | 10 |
| $\mathrm{Arg}_1(d_2 \to d_3; 1)$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $-\frac{2}{7}$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | | $-$ |
| $-6\beta c'^2 \equiv c' \pmod 7$ | 2 | 4 | 1 | 3 | 2 | 6 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{c'}{7}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{6}{7}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 2)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | $-$ | | |
| $3\beta c'$ (mod 14) | 10 | 10 | 3 | 3 | 10 | 10 |
| $-27\beta c'^2$ (mod 14) | 2 | 4 | 1 | 3 | 2 | 6 |
| $\mathrm{Arg}_1(d_2 \to d_3; 3)$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{5}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_2 \to d_3; 3)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 3)$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{5}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $-$ | | | | $-$ | |

Table 8.14: Table for $\mathrm{Arg}(d_2 \to d_3; \ell)$ in (8.45); $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

So $-84s(d_4,c)+84s(d_3,c) \pmod{168}$ is uniquely determined by $12 \pmod{24}$ and $2\beta \pmod 7$ and

$$\mathrm{Arg}_2(d_3\to d_4;\ell)=\frac{1,3,5,9,11,13}{14}\quad \text{when } \beta=6,4,2,5,3,1, \text{ resp.}$$

Moreover, $\mathrm{Arg}_3(d_3\to d_4;\ell)=\frac{5\beta}{7}$. This gives Table 8.15.

Now we have finished the proof for $2\nmid c'$, $3\nmid c'$ and $7\nmid c'$ by comparing Table 8.13, Table 8.14, and Table 8.15 with Condition 8.3.

### 8.2.2 $2\nmid c', 3|c', 7\nmid c'$

In this case $c'\equiv 3,9,15,27,33,39 \pmod{42}$. First we check $\mathrm{Arg}(d_1\to d_2;\ell)$ with $d_1a_1\equiv d_2a_4\equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_1\to d_2;\ell)=-\frac{9\beta c'^2\ell^2}{14}\begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_4\ell}{7})/\sin(\frac{\pi a_1\ell}{7}))=1,\\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_4\ell}{7})/\sin(\frac{\pi a_1\ell}{7}))=-1.\end{cases}$$

When $\ell=1$, the sign changes when $3\beta c'\equiv 10 \pmod{14}$. When $\ell=2$, the sign always changes. When $\ell=3$, the sign changes when $9\beta c'\equiv 9 \pmod{14}$ but keeps when $9\beta c'\equiv 2 \pmod{14}$.

We have $\theta=3$, $6cs(d,c)\in\mathbb{Z}$, and

$$-12cs(d_2,c)+12cs(d_1,c)\equiv -d_2-\overline{d_{2\{3c\}}}+d_1+\overline{d_{1\{3c\}}}\equiv -\beta c'+\beta c'\overline{d_{1\{3c\}}}\cdot\overline{d_{2\{3c\}}} \pmod{3c}.$$

Here $\overline{d_{1\{3c\}}}$ is the inverse of $d_1 \pmod{3c}$ and we have used (8.12). Hence we confirm that $-12cs(d_2,c)+12cs(d_1,c)$ is a multiple of $c'$. After dividing the above congruence by $c'$, we obtain a congruence modulo 21 while $\overline{d_{j\{3c\}}}\equiv a_{\overline{j\{7\}}} \pmod{21}$ due to $21|c$. Hence

$$-84s(d_2,c)+84s(d_1,c)\equiv -\beta+\beta a_1a_4\equiv \beta(a_1a_4-1) \pmod{21}.$$

We have $a_1a_4\equiv 4 \pmod{21}$ by $a_4a_1\equiv 1 \pmod 3$ and $a_1a_4\equiv 4 \pmod 7$. Hence

$$-28s(d_2,c)+28s(d_1,c)\equiv\beta \pmod 7.$$

Due to $(\frac{2}{7})=1$, we also have

$$-12cs(d_1,c)+12cs(d_2,c)\equiv 2(\tfrac{d_1}{7})(\tfrac{d_1}{c'})-2(\tfrac{d_2}{7})(\tfrac{d_2}{c'})\equiv 0 \pmod 8.$$

Since $3c'$ is odd, we still have $-28s(d_2,c)+28s(d_1,c)\equiv 0 \pmod 8$. Now we get $\mathrm{Arg}_2(d_1\to d_2;\ell)=\frac{\overline{8_{\{21\}}}\cdot\beta}{7}=\frac{\beta}{7}$ and $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1\to d_2;\ell)=\frac{-\beta}{7}$. This gives Table 8.16.

Next we investigate $\mathrm{Arg}(d_2\to d_3;\ell)$ with $d_2a_4\equiv d_3a_5\equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_2\to d_3;\ell)=-\frac{3\beta c'^2\ell^2}{14}\begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_5\ell}{7})/\sin(\frac{\pi a_4\ell}{7}))=1,\\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_5\ell}{7})/\sin(\frac{\pi a_4\ell}{7}))=-1.\end{cases}$$

When $\ell=1$, the sign changes when $\beta c'\equiv 8 \pmod{14}$. When $\ell=2$, the sign remains. When $\ell=3$, the sign changes when $3\beta c'\equiv 3 \pmod{14}$ but remains when $3\beta c'\equiv 10 \pmod{14}$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 1 | 5 | 11 | 13 | 17 | 19 |
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $3\beta c'$ (mod 14) | 3 | 3 | 10 | 10 | 3 | 3 |
| $9\beta c'^2$ (mod 14) | 9 | 3 | 8 | 12 | 13 | 3 |
| $\mathrm{Arg}_1(d_3 \to d_4; 1)$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{8}{14}+\frac{1}{2}$ | $\frac{12}{14}+\frac{1}{2}$ | $\frac{13}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 1)$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $+$ | | | $-$ | | |
| $18\beta c'^2 \equiv 4c' \pmod 7$ | 4 | 6 | 2 | 3 | 5 | 6 |
| $\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2}+\frac{4c'}{7}$ | $\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 2)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 2)$ | $-\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | $-$ | |
| $9\beta c'$ (mod 14) | 9 | 9 | 2 | 2 | 9 | 9 |
| $81\beta c'^2$ (mod 14) | 11 | 13 | 2 | 10 | 5 | 13 |
| $\mathrm{Arg}_1(d_3 \to d_4; 3)$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{13}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 3)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 3)$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | | | $+$ |
| $c'$ (mod 42) | 23 | 25 | 29 | 31 | 37 | 41 |
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $3\beta c'$ (mod 14) | 10 | 10 | 3 | 3 | 10 | 10 |
| $9\beta c'^2$ (mod 14) | 4 | 8 | 9 | 13 | 4 | 12 |
| $\mathrm{Arg}_1(d_3 \to d_4; 1)$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 1)$ | $-\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | | $-$ |
| $18\beta c'^2 \equiv 4c' \pmod 7$ | 1 | 2 | 4 | 5 | 1 | 3 |
| $\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2}+\frac{3c'}{7}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 2)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 2)$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | $-$ | | |
| $9\beta c'$ (mod 14) | 2 | 2 | 9 | 9 | 2 | 2 |
| $-81\beta c'^2$ (mod 14) | 8 | 2 | 11 | 5 | 8 | 10 |
| $\mathrm{Arg}_1(d_3 \to d_4; 3)$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3 \to d_4; 3)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 3)$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $-$ | | | | $-$ | |

Table 8.15: Table for $\mathrm{Arg}(d_3 \to d_4; \ell)$ in (8.45); $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 3 | 9 | 15 | 27 | 33 | 39 |
| $\beta$ | 5 | 4 | 1 | 6 | 3 | 2 |
| $3\beta c'$ (mod 14) | 3 | 10 | 3 | 10 | 3 | 10 |
| $-9\beta c'^2$ (mod 14) | 1 | 10 | 5 | 2 | 11 | 6 |
| $\mathrm{Arg}_1(d_1 \to d_2; 1)$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{11}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | $-$ | | |
| $-18\beta c'^2 \equiv 3c' \pmod 7$ | 2 | 6 | 3 | 4 | 1 | 5 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $-\frac{3}{14}$ | $\frac{3}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $-$ | | | | | $+$ |
| $9\beta c'$ (mod 14) | 9 | 2 | 9 | 2 | 9 | 2 |
| $-81\beta c'^2$ (mod 14) | 9 | 6 | 3 | 4 | 1 | 12 |
| $\mathrm{Arg}_1(d_1 \to d_2; 3)$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 3)$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $-$ | | | $+$ | |

Table 8.16: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2 \nmid c$, $3|c$, $7 \nmid c$.

We have $\theta = 3$, $6cs(d, c) \in \mathbb{Z}$, and

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv -d_3 - \overline{d_{3\{3c\}}} + d_2 + \overline{d_{2\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{2\{3c\}}} \cdot \overline{d_{3\{3c\}}} \pmod{3c}.$$

Hence we confirm that $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of $c'$. After dividing by $c'$, we obtain a congruence modulo 21 and

$$-84s(d_3, c) + 84s(d_2, c) \equiv -\beta + \beta a_4 a_5 \equiv \beta(a_4 a_5 - 1) \pmod{21}.$$

Since $a_4 a_5 \equiv 13 \pmod{21}$ by $a_4 a_5 \equiv 1 \pmod 3$ and $a_4 a_5 \equiv -1 \pmod 7$, we have

$$-28s(d_3, c) + 28s(d_2, c) \equiv 4\beta \pmod 7.$$

By (8.11), we get

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) - 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) \equiv 4 \pmod 8.$$

Since $3c'$ is odd, we still have $-28s(d_3, c) + 28s(d_2, c) \equiv 4 \pmod 8$. Now $4\beta \pmod 7$ and $4 \pmod 8$ determines a unique residue modulo 56 and then

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) \equiv \frac{1, 3, 5, 9, 11, 13}{14} \pmod 1 \quad \text{when} \quad \beta = 1, 3, 5, 2, 4, 6.$$

This gives Table 8.17.

| $c'$ (mod 42) | 3 | 9 | 15 | 27 | 33 | 39 |
|---|---|---|---|---|---|---|
| $\beta$ | 5 | 4 | 1 | 6 | 3 | 2 |
| $\beta c'$ (mod 14) | 1 | 8 | 1 | 8 | 1 | 8 |
| $-3\beta c'^2$ (mod 14) | 5 | 8 | 11 | 10 | 13 | 2 |
| $\mathrm{Arg}_1(d_2 \to d_3; 1)$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $-\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 1)$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | $-$ | | |
| $-6\beta c'^2 \equiv c'$ (mod 7) | 3 | 2 | 1 | 6 | 5 | 4 |
| $\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{6}{7}$ | $\frac{5}{7}$ | $\frac{4}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | | $+$ |
| $3\beta c'$ (mod 14) | 3 | 10 | 3 | 10 | 3 | 10 |
| $-27\beta c'^2$ (mod 14) | 3 | 2 | 1 | 6 | 5 | 4 |
| $\mathrm{Arg}_1(d_1 \to d_2; 3)$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 3)$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $-$ | | | $+$ | |

Table 8.17: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2 \nmid c$, $3 | c$, $7 \nmid c$.

Finally we deal with $\mathrm{Arg}(d_3 \to d_4; \ell)$ where $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_3 \to d_4; \ell) = \frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = -1. \end{cases}$$

When $\ell = 1$, the sign changes when $3\beta c' \equiv 10 \pmod{14}$ but remains when $3\beta c' \equiv 3 \pmod{14}$. When $\ell = 2$, the sign always changes. When $\ell = 3$, the sign changes when $9\beta c' \equiv 2 \pmod{14}$ but remains when $9\beta c' \equiv 9 \pmod{14}$.

We have $\theta = 3$, $6cs(d, c) \in \mathbb{Z}$, and

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv -d_4 - \overline{d_{4\{3c\}}} + d_3 + \overline{d_{3\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{3\{3c\}}} \cdot \overline{d_{4\{3c\}}} \pmod{3c}.$$

We again confirm that $-12cs(d_4, c) + 12cs(d_3, c)$ is a multiple of $c'$. After dividing the above congruence by $c'$, we obtain a congruence modulo 21 and

$$-84s(d_4, c) + 84s(d_3, c) \equiv -\beta + \beta a_5 a_2 \equiv \beta(a_5 a_2 - 1) \pmod{21}.$$

Since $a_2 a_5 \equiv 10 \pmod{21}$ by$a_5 a_2 \equiv 1 \pmod 3$ and $a_5 a_2 \equiv 3 \pmod 7$, we get

$$-28s(d_4, c) + 28s(d_3, c) \equiv 3\beta \pmod 7.$$

We also have

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv 2\left(\tfrac{d_4}{7}\right)\left(\tfrac{d_4}{c'}\right) - 2\left(\tfrac{d_3}{7}\right)\left(\tfrac{d_3}{c'}\right) \equiv 4 \pmod 8.$$

Since $3c'$ is odd, we get $-28s(d_4,c)+28s(d_3,c)\equiv 4 \pmod 8$. Now $3\beta \pmod 7$ and $4 \pmod 8$ determines a unique residue modulo 56 and then

$$\mathrm{Arg}_2(d_2\to d_3;\ell)\equiv \frac{1,3,5,9,11,13}{14} \pmod 1 \quad \text{when} \quad \beta=6,4,2,5,3,1.$$

This gives Table 8.18 and we have finished the proof for $c'\equiv 3,9,15,27,33,39 \pmod{42}$.

| $c' \pmod{42}$ | 3 | 9 | 15 | 27 | 33 | 39 |
|---|---|---|---|---|---|---|
| $\beta$ | 5 | 4 | 1 | 6 | 3 | 2 |
| $3\beta c' \pmod{14}$ | 3 | 10 | 3 | 10 | 3 | 10 |
| $9\beta c'^2 \pmod{14}$ | 13 | 4 | 9 | 12 | 3 | 8 |
| $\mathrm{Arg}_1(d_3\to d_4;1)$ | $\frac{13}{14}$ | $-\frac{3}{14}$ | $\frac{9}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3\to d_4;1)$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ |
| Total $\mathrm{Arg}(d_1\to d_2;1)$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ |
| $c'\ell\equiv\pm1 \pmod 7$? | | | $+$ | $-$ | | |
| $18\beta c'^2\equiv 4c' \pmod 7$ | 5 | 1 | 4 | 3 | 6 | 2 |
| $\mathrm{Arg}_1(d_3\to d_4;2):\frac{1}{2}+\frac{4c'}{7}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ | $\frac{1}{14}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3\to d_4;1)$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ |
| Total $\mathrm{Arg}(d_3\to d_4;2)$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ |
| $c'\ell\equiv\pm1 \pmod 7$? | $-$ | | | | | $+$ |
| $9\beta c' \pmod{14}$ | 9 | 2 | 9 | 2 | 9 | 2 |
| $81\beta c'^2 \pmod{14}$ | 5 | 8 | 11 | 10 | 13 | 2 |
| $\mathrm{Arg}_1(d_1\to d_2;3)$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $-\frac{5}{14}$ |
| $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_3\to d_4;1)$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ |
| Total $\mathrm{Arg}(d_1\to d_2;3)$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ |
| $c'\ell\equiv\pm1 \pmod 7$? | | $-$ | | | $+$ | |

Table 8.18: Table for $\mathrm{Arg}(d_3\to d_4;\ell)$ in (8.45); $2\nmid c$, $3|c$, $7\nmid c$.

### 8.2.3 $2|c', 3\nmid c', 7\nmid c'$

In this case $c'\equiv 2,4,8,10,16,20,22,26,32,34,38,40 \pmod{42}$. First we deal with $\mathrm{Arg}(d_1\to d_2;\ell)$:

$$\mathrm{Arg}_1(d_1\to d_2;\ell)=-\frac{9\beta c'^2\ell^2}{14}\begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_4\ell}{7})/\sin(\frac{\pi a_1\ell}{7}))=1,\\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_4\ell}{7})/\sin(\frac{\pi a_1\ell}{7}))=-1.\end{cases}$$

Now $c'$ is even, so $\beta c'\equiv 8 \pmod{14}$. When $\ell=1,2$, the sign changes. When $\ell=3$, the sign remains.

For $\mathrm{Arg}_2$ we need to combine (8.9) and (8.11). We have $12cs(d,c)\equiv 0 \pmod 6$ and

$$-12cs(d_2,c)+12cs(d_1,c)\equiv -d_2-a_4+d_1+a_1\equiv -4\beta c' \pmod c. \tag{8.47}$$

Then $-12cs(d_2,c)+12cs(d_1,c)$ is a multiple of $c'$.

We claim that

$$-12cs(d_2,c)+12cs(d_1,c)\equiv 0 \pmod{8\times 2^{\lambda}}. \tag{8.48}$$

Denote $\lambda \geq 1$ by $2^\lambda \| c$. We have

$$\begin{aligned}-12cs(d_2,c)+12cs(d_1,c) \equiv& -d_2-\overline{d_{2\{8\times 2^\lambda\}}}(c^2+3c+1+2c(\tfrac{c}{d_2}))\\ &+d_1+\overline{d_{1\{8\times 2^\lambda\}}}(c^2+3c+1+2c(\tfrac{c}{d_1}))\\ \equiv& -\beta c'+\beta c'\overline{d_{2\{8\times 2^\lambda\}}}\cdot\overline{d_{1\{8\times 2^\lambda\}}}(c^2+3c+1)\\ &+2c(\overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})-\overline{d_{2\{8\times 2^\lambda\}}}(\tfrac{c}{d_2})) \pmod{8\times 2^\lambda}.\end{aligned}$$

After dividing $c'$, we get the value modulo 8 by $\overline{x_{\{8\}}} \equiv x \pmod 8$:

$$\begin{aligned}-84s(d_2,c)+84s(d_1,c) &\equiv -\beta+\beta d_2 d_1(c^2+3c+1)+6(d_1(\tfrac{c}{d_1})-d_2(\tfrac{c}{d_2}))\\ &\equiv \beta c'(1+d_1\beta)(c'+1)-2(d_1(\tfrac{c}{d_1})-d_2(\tfrac{c}{d_2})) \pmod 8\end{aligned}$$

For the first value val.:= $\beta c'(1+d_1\beta)(c'+1) \pmod 8$, we see that $\beta(1+d_1\beta)$ is even (becuase $d_1$ is odd) and $c'$ is even, hence the result is $0,4 \pmod 8$. Moreover, val. is the same for $c'$ and $c'+7$. Then we have Table 8.19.

| $c' \pmod 7$<br>$\beta$<br>$\beta c'$<br>$\beta d_1+1$ | 1<br>1<br>$c'$<br>$d_1+1$ | 2<br>4<br>$4c'$<br>$4d_1+1$ | 3<br>5<br>$5c'$<br>$5d_1+1$ | 4<br>2<br>$2c'$<br>$2d_1+1$ | 5<br>3<br>$3c'$<br>$3d_1+1$ | 6<br>6<br>$6c'$<br>$6d_1+1$ |
|---|---|---|---|---|---|---|
| $2\|c$, $d_1 \equiv 1 \pmod 4$ | 4 | 0 | 4 | 4 | 0 | 4 |
| $2\|c$, $d_1 \equiv 3 \pmod 4$ | 0 | 0 | 0 | 4 | 4 | 4 |
| $4\mid c$ | 0 | 0 | 0 | 0 | 0 | 0 |

Table 8.19: Table for val.:= $\beta c'(\beta d_1+1)(c'+1) \pmod 8$; $2\mid c$, no requirement for $(c,3)$, $7 \nmid c$.

For the other part we determine whether

$$d_1(\tfrac{c}{d_1})-d_2(\tfrac{c}{d_2}) \equiv 0 \text{ or } 2 \pmod 4. \tag{8.49}$$

When $2^\lambda \| c$ and $\lambda \geq 2$ is even, we have $d_1 \equiv d_2 \pmod 4$ and $(\frac{2^\lambda}{d})=1$, hence (8.49) is $0 \pmod 4$. When $\lambda \geq 3$ is odd, then $d_2 \equiv d_1 \pmod 8$ and we still have $(\frac{2^\lambda}{d_1})=(\frac{2^\lambda}{d_2})$. Then when $4\mid c$, we get (8.49) always divisible by 4, which matches the last row of Table 8.19.

When $2\|c$, by $(\frac{7}{x})=(\frac{x}{7})(-1)^{\frac{x-1}{2}}$ for odd $x$, we have

$$d_1(\tfrac{c}{d_1})-d_2(\tfrac{c}{d_2}) \equiv (\tfrac{d_1}{c'/2})\left((-1)^{\frac{d_1-1}{2}+\frac{d_1-1}{2}\cdot\frac{\frac{c'}{2}-1}{2}}(\tfrac{2}{d_1})d_1-(-1)^{\frac{d_2-1}{2}+\frac{d_2-1}{2}\cdot\frac{\frac{c'}{2}-1}{2}}(\tfrac{2}{d_2})d_2\right) \pmod 4. \tag{8.50}$$

Since $d_2=d_1+\beta c'$, we divide into cases for $c' \equiv 2,6 \pmod 8$, $d_1 \equiv 1,3,5,7 \pmod 8$ and $\beta$ from 1 to 6 to make Table 8.20. Note that $d_2 \pmod 8$ is derived by $c' \pmod 8$, $\beta$ and $d_1 \pmod 8$.

Comparing Table 8.19 and Table 8.20, we have proved (8.48). Combining (8.47) and $12cs(d,c) \equiv 0 \pmod 6$, we divide $24c'$ to compute $\mathrm{Arg}_2(d_1 \to d_2;\ell)=\frac{\beta}{7}$. Then $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2;\ell)=-\frac{\beta}{7}$ and we have Table 8.21.

Next we deal with $\mathrm{Arg}(d_2 \to d_3;\ell)$ with $d_2a_4 \equiv d_3a_5 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_2 \to d_3;\ell) = -\frac{3\beta c'^2\ell^2}{14}\begin{cases}+0 & \mathrm{sgn}(\sin(\frac{\pi a_5\ell}{7})/\sin(\frac{\pi a_4\ell}{7}))=1,\\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_5\ell}{7})/\sin(\frac{\pi a_4\ell}{7}))=-1.\end{cases}$$

| (8.49) $\searrow$ | $c' \equiv 2 \pmod 8$ | | | | $c' \equiv 6 \pmod 8$ | | | |
|---|---|---|---|---|---|---|---|---|
| $d_1 \pmod 8$ | 1 | 3 | 5 | 7 | 1 | 3 | 5 | 7 |
| $\beta = 1$, (8.49) | 2 | 0 | 2 | 0 | 2 | 0 | 2 | 0 |
| $\beta = 4$, (8.49) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\beta = 5$, (8.49) | 2 | 0 | 2 | 0 | 2 | 0 | 2 | 0 |
| $\beta = 2$, (8.49) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| $\beta = 3$, (8.49) | 0 | 2 | 0 | 2 | 0 | 2 | 0 | 2 |
| $\beta = 6$, (8.49) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 8.20: Table for (8.49); $2|c$, no requirement for $(c, 3)$, $7 \nmid c$.

Now $c'$ is even, so $\beta c' \equiv 8 \pmod{14}$. When $\ell = 1$, the sign changes. When $\ell = 2, 3$, the sign remains.

For $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ we do the similar proof as $\mathrm{Arg}_2(d_1 \to d_2; \ell)$. First we have

$$- 12cs(d_3, c) + 12cs(d_2, c) \equiv -d_3 - a_5 + d_2 + a_4 \equiv -2\beta c' \pmod c. \tag{8.51}$$

Then $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of $c'$.

We claim that

$$- 12cs(d_3, c) + 12cs(d_2, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}. \tag{8.52}$$

For $\lambda \geq 1$ such that $2^\lambda \| c$, we have

$$\begin{aligned} -12cs(d_3, c) + 12cs(d_2, c) \equiv & - d_3 - \overline{d_{3\{8\times 2^\lambda\}}}(c^2 + 3c + 1 + 2c(\tfrac{c}{d_3})) \\ & + d_2 + \overline{d_{2\{8\times 2^\lambda\}}}(c^2 + 3c + 1 + 2c(\tfrac{c}{d_2})) \\ \equiv & - \beta c' + \beta c' \overline{d_{3\{8\times 2^\lambda\}}} \cdot \overline{d_{2\{8\times 2^\lambda\}}}(c^2 + 3c + 1) \\ & + 2c(\overline{d_{2\{8\times 2^\lambda\}}}(\tfrac{c}{d_2}) - \overline{d_{3\{8\times 2^\lambda\}}}(\tfrac{c}{d_3})) \pmod{8 \times 2^\lambda}, \end{aligned}$$

After dividing $c'$, since $2^\lambda \| c'$ and $\overline{x_{\{8\}}} \equiv x \pmod 8$ for odd $x$, we have

$$\begin{aligned} -84s(d_3, c) + 84s(d_2, c) &\equiv -\beta + \beta d_3 d_2 (c^2 + 3c + 1) + 6(d_2(\tfrac{c}{d_2}) - d_3(\tfrac{c}{d_3})) \\ &\equiv \beta c'(1 + d_2\beta)(c' + 1) - 2(d_2(\tfrac{c}{d_2}) - d_3(\tfrac{c}{d_3})) \pmod 8 \end{aligned}$$

The proof of (8.52) is then the same as the proof of (8.48) before, noting that in the second part we have $(\frac{d_2}{7}) = 1$ while $(\frac{d_3}{7}) = -1$. This difference makes an alternation in Table 8.20 where we should change all 2 to 0 and all 0 to 2, which results in $4 \times 2^\lambda \pmod{8 \times 2^\lambda}$ rather than $0 \pmod{8 \times 2^\lambda}$ in (8.52). We omit the details.

Combining (8.51), (8.52) and $12cs(d, c) \equiv 0 \pmod 6$ we can determine $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ with denominator 42 and numerator by $3\beta \pmod 7$ and $3 \pmod 6$, hence

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 1, 3, 5, 2, 4, 6, \text{ resp.}$$

Now we have Table 8.22.

Finally we check $\mathrm{Arg}(d_2 \to d_3; \ell)$ with $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_2 \to d_3; \ell) = \frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = -1. \end{cases}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 2 | 4 | 8 | 10 | 16 | 20 |
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $-3\beta c'^2$ (mod 14) | 10 | 6 | 12 | 8 | 10 | 2 |
| $\text{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{4}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{6}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 1)$ | $-\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | | | $-$ |
| $-18\beta c'^2 \equiv 3c'$ (mod 7) | 6 | 5 | 3 | 2 | 6 | 4 |
| $\text{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{6}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 2)$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | $-$ | | |
| $-81\beta c'^2$ (mod 14) | 6 | 12 | 10 | 2 | 6 | 4 |
| $\text{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$ | $\frac{3}{7}$ | $\frac{6}{7}$ | $\frac{5}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{4}{7}$ | $-\frac{2}{7}$ | $-\frac{1}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{6}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 3)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | $-$ | |
| $c'$ (mod 42) | 22 | 26 | 32 | 34 | 38 | 40 |
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $-9\beta c'^2$ (mod 14) | 12 | 4 | 6 | 2 | 8 | 4 |
| $\text{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{6}{7}$ | $-\frac{5}{7}$ | $-\frac{3}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 1)$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{5}{14}$ |
| $c'\ell = \pm 1$ (mod 7)? | $+$ | | | $-$ | | |
| $-18\beta c'^2 \equiv 3c'$ (mod 7) | 3 | 1 | 5 | 4 | 2 | 1 |
| $\text{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $-\frac{5}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{6}{7}$ | $-\frac{5}{7}$ | $-\frac{3}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 2)$ | $-\frac{3}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | | $-$ | |
| $-81\beta c'^2$ (mod 14) | 10 | 8 | 12 | 4 | 2 | 8 |
| $\text{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$ | $\frac{5}{7}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{4}{7}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{6}{7}$ | $-\frac{5}{7}$ | $-\frac{3}{7}$ |
| Total $\text{Arg}(d_1 \to d_2; 3)$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{3}{7}$ | $\frac{1}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | | | $+$ |

Table 8.21: Table for $\text{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2|c$, $3 \nmid c$, $7 \nmid c$.

| $c'$ (mod 42) | 2 | 4 | 8 | 10 | 16 | 20 |
|---|---|---|---|---|---|---|
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $-3\beta c'^2$ (mod 14) | 8 | 2 | 4 | 12 | 8 | 10 |
| $\mathrm{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 1)$ | $-\frac{2}{7}$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | | $-$ |
| $-6\beta c'^2 \equiv c' \pmod 7$ | 2 | 4 | 1 | 3 | 2 | 6 |
| $\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{3c'}{7}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{6}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | $-$ | | |
| $-27\beta c'^2$ (mod 14) | 2 | 4 | 8 | 10 | 2 | 6 |
| $\mathrm{Arg}_1(d_2 \to d_3; 3) : -\frac{27\beta c'^2}{14}$ | $\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{5}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$ | $\frac{9}{14}$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 3)$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{5}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $-$ | | | | $-$ | |

| $c'$ (mod 42) | 22 | 26 | 32 | 34 | 38 | 40 |
|---|---|---|---|---|---|---|
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $-3\beta c'^2$ (mod 14) | 4 | 6 | 2 | 10 | 12 | 6 |
| $\mathrm{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 1)$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{2}{7}$ |
| $c'\ell = \pm 1 \pmod 7$? | $+$ | | | $-$ | | |
| $-6\beta c'^2 \equiv c' \pmod 7$ | 1 | 5 | 4 | 6 | 3 | 5 |
| $\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{3c'}{7}$ | $\frac{1}{7}$ | $\frac{5}{7}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | | $-$ | |
| $-27\beta c'^2$ (mod 14) | 8 | 12 | 4 | 6 | 10 | 12 |
| $\mathrm{Arg}_1(d_2 \to d_3; 3) : -\frac{27\beta c'^2}{14}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ | $\frac{6}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 3)$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | | | $+$ |

Table 8.22: Table for $\mathrm{Arg}(d_2 \to d_3; \ell)$ in (8.45); $2|c$, $3 \nmid c$, $7 \nmid c$.

Since $c'$ is even, we have $-3\beta c' \equiv 4 \pmod{14}$ and the sign always changes.

For $\mathrm{Arg}_2(d_3 \to d_4; \ell)$ we argue as for $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ and $\mathrm{Arg}_2(d_1 \to d_2; \ell)$. First we have

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv -d_4 - a_2 + d_3 + a_5 \equiv 2\beta c' \pmod{c}. \tag{8.53}$$

Then $-12cs(d_4, c) + 12cs(d_3, c)$ is a multiple of $c'$. We claim that

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}. \tag{8.54}$$

The proof is the same as the proof for (8.52) and we omit the details. Combining (8.53), (8.54) and (8.8), we can determine $\mathrm{Arg}_2(d_3 \to d_4; \ell)$ with denominator 42 and numerator by $4\beta \pmod 7$ and $3 \pmod 6$, hence

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 6, 4, 2, 5, 3, 1.$$

This gives Table 8.23.

Comparing Tables 8.21, 8.22 and 8.23 we see that when $2|c'$, $3 \nmid c'$ and $7 \nmid c'$, Condition 8.3 holds and we have proved (8.45).

### 8.2.4 $2|c', 3|c', 7 \nmid c'$

In this case $c' \equiv 6, 12, 18, 24, 30, 36 \pmod{42}$. First we deal with $\mathrm{Arg}(d_1 \to d_2; \ell)$:

$$\mathrm{Arg}_1(d_1 \to d_2; \ell) = -\frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_4 \ell}{7}) / \sin(\frac{\pi a_1 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_4 \ell}{7}) / \sin(\frac{\pi a_1 \ell}{7})) = -1. \end{cases}$$

Now $c'$ is even, so $\beta c' \equiv 8 \pmod{14}$. When $\ell = 1, 2$, the sign changes. When $\ell = 3$, the sign remains.

For $\mathrm{Arg}_2$ we need to combine (8.8) and (8.11). We have $12cs(d, c) \equiv 0 \pmod 2$ and

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv -d_2 - \overline{d_{2\{3c\}}} + d_1 + \overline{d_{1\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{2\{3c\}}} \cdot \overline{d_{1\{3c\}}} \pmod{3c}. \tag{8.55}$$

Then $-12cs(d_2, c) + 12cs(d_1, c)$ is a multiple of $c'$. After dividing $c'$, since $3|c'$ and $\overline{d_{j\{3c\}}} \equiv a_{\overline{j_{\{7\}}}} \pmod{21}$, we get

$$-84s(d_2, c) + 84s(d_1, c) \equiv -\beta + \beta a_4 a_1 \equiv 3\beta \pmod{21}. \tag{8.56}$$

where the last congruence equality follows since $a_4 a_1 \equiv 4 \pmod 7$ and $a_4 a_1 \equiv 1 \pmod 3$.

We still have (8.48):

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv 0 \pmod{8 \times 2^\lambda}$$

because the proof of (8.48) does not depend on whether $3|c$ or not. Combining the above two congruences we have $\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{\beta}{7}$. Then $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; \ell) = -\frac{\beta}{7}$, which gives Table 8.24.

Next we check $\mathrm{Arg}(d_2 \to d_3; \ell)$ with $d_2 a_4 \equiv d_3 a_5 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_2 \to d_3; \ell) = -\frac{3\beta c'^2 \ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7}) / \sin(\frac{\pi a_4 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7}) / \sin(\frac{\pi a_4 \ell}{7})) = -1. \end{cases}$$

Now $c'$ is even, so $\beta c' \equiv 8 \pmod{14}$. When $\ell = 1$, the sign changes. When $\ell = 2, 3$, the sign remains.

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 2 | 4 | 8 | 10 | 16 | 20 |
| $\beta$ | 4 | 2 | 1 | 5 | 4 | 6 |
| $9\beta c'^2$ (mod 14) | 4 | 8 | 2 | 6 | 4 | 12 |
| $\mathrm{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 1)$ | $-\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | | | $-$ |
| $18\beta c'^2 \equiv 4c'$ (mod 7) | 1 | 2 | 4 | 5 | 1 | 3 |
| $\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | $-$ | | |
| $81\beta c'^2$ (mod 14) | 8 | 2 | 4 | 12 | 8 | 10 |
| $\mathrm{Arg}_1(d_3 \to d_4; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{1}{14}$ | $\frac{11}{14}$ | $\frac{9}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 3)$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | $-$ | |
| $c'$ (mod 42) | 22 | 26 | 32 | 34 | 38 | 40 |
| $\beta$ | 1 | 3 | 2 | 6 | 5 | 3 |
| $9\beta c'^2$ (mod 14) | 2 | 10 | 8 | 12 | 6 | 10 |
| $\mathrm{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $\frac{3}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_3 \to d_4; 1)$ | $\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $\frac{1}{7}$ |
| $c'\ell = \pm 1$ (mod 7)? | $+$ | | | $-$ | | |
| $18\beta c'^2 \equiv 4c'$ (mod 7) | 4 | 6 | 2 | 3 | 5 | 6 |
| $\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$ | $\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 2)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 2)$ | $-\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | | $-$ | |
| $81\beta c'^2$ (mod 14) | 4 | 6 | 2 | 10 | 12 | 6 |
| $\mathrm{Arg}_1(d_2 \to d_3; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{9}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{13}{14}$ |
| Total $\mathrm{Arg}(d_2 \to d_3; 3)$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{1}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | | | $+$ |

Table 8.23: Table for $\mathrm{Arg}(d_3 \to d_4; \ell)$ in (8.45); $2|c$, $3 \nmid c$, $7 \nmid c$.

| $c'$ (mod 42) <br> $\beta$ <br> $-9\beta c'^2$ (mod 14) | 6 <br> 6 <br> 2 | 12 <br> 3 <br> 4 | 18 <br> 2 <br> 6 | 24 <br> 5 <br> 8 | 30 <br> 4 <br> 10 | 36 <br> 1 <br> 12 |
|---|---|---|---|---|---|---|
| $\mathrm{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 1)$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | $-$ | | | | | $+$ |
| $-18\beta c'^2 \equiv 3c' \pmod 7$ | 4 | 1 | 5 | 2 | 6 | 3 |
| $\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 2)$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 2)$ | $\frac{3}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ | $-\frac{3}{14}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | | $+$ | $-$ | | |
| $-81\beta c'^2$ (mod 14) | 4 | 8 | 12 | 2 | 6 | 10 |
| $\mathrm{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$ | $\frac{2}{7}$ | $\frac{4}{7}$ | $\frac{6}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ | $\frac{5}{7}$ |
| $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 2)$ | $-\frac{6}{7}$ | $-\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{5}{7}$ | $-\frac{4}{7}$ | $-\frac{1}{7}$ |
| Total $\mathrm{Arg}(d_1 \to d_2; 3)$ | $\frac{3}{7}$ | $\frac{1}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{1}{7}$ | $-\frac{3}{7}$ |
| $c'\ell \equiv \pm 1 \pmod 7$? | | $+$ | | | $-$ | |

Table 8.24: Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2|c$, $3|c$, $7 \nmid c$.

For $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ we have

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv -d_3 - \overline{d_{3\{3c\}}} + d_2 + \overline{d_{2\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{3\{3c\}}} \cdot \overline{d_{2\{3c\}}} \pmod c. \tag{8.57}$$

Then $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of $c'$. After dividing by $c'$ we get

$$-84s(d_3, c) + 84s(d_2, c) \equiv -\beta + \beta a_5 a_4 \equiv 12\beta \pmod{21} \tag{8.58}$$

where the last congruence is by $a_5 a_4 \equiv 20 \pmod 7$ and $a_5 a_4 \equiv 1 \pmod 3$.

The equality (8.52) still holds:

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}$$

because its proof does not involve whether $3|c'$ or not. Combining the two congruences above we can decide $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ via $4\beta \pmod 7$ and $4 \pmod 8$:

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 1, 3, 5, 2, 4, 6.$$

This gives Table 8.25.

Finally we check $\mathrm{Arg}(d_3 \to d_4; \ell)$ with $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$:

$$\mathrm{Arg}_1(d_3 \to d_4; \ell) = \frac{9\beta c'^2 \ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_2 \ell}{7}) / \sin(\frac{\pi a_5 \ell}{7})) = -1. \end{cases}$$

Since $c'$ is even, we have $-3\beta c' \equiv 4 \pmod{14}$ and the sign always changes.

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 6 | 12 | 18 | 24 | 30 | 36 |
| $\beta$ | 6 | 3 | 2 | 5 | 4 | 1 |
| $-3\beta c'^2$ (mod 14) | 10 | 6 | 2 | 12 | 8 | 4 |
| $\text{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_2 \to d_3; 1)$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ |
| Total $\text{Arg}(d_2 \to d_3; 1)$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{2}{7}$ | $-\frac{3}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | | $+$ |
| $-6\beta c'^2 \equiv c'$ (mod 7) | 6 | 5 | 4 | 3 | 2 | 1 |
| $\text{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$ | $\frac{6}{7}$ | $\frac{5}{7}$ | $\frac{4}{7}$ | $\frac{3}{7}$ | $\frac{2}{7}$ | $\frac{1}{7}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_2 \to d_3; 2)$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ |
| Total $\text{Arg}(d_2 \to d_3; 2)$ | $\frac{1}{14}$ | $\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{5}{14}$ | $-\frac{1}{14}$ | $-\frac{1}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | $-$ | | |
| $-27\beta c'^2$ (mod 14) | 6 | 12 | 4 | 10 | 2 | 8 |
| $\text{Arg}_1(d_1 \to d_2; 3) : -\frac{27\beta c'^2}{14}$ | $\frac{3}{7}$ | $\frac{6}{7}$ | $\frac{2}{7}$ | $\frac{5}{7}$ | $\frac{1}{7}$ | $\frac{4}{7}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_2 \to d_3; 3)$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $\frac{13}{14}$ | $\frac{9}{14}$ | $\frac{11}{14}$ |
| Total $\text{Arg}(d_2 \to d_3; 3)$ | $-\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{5}{14}$ | $-\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{5}{14}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | | $-$ | |

Table 8.25: Table for $\text{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2|c$, $3|c$, $7 \nmid c$.

For $\text{Arg}_2(d_3 \to d_4; \ell)$, first we have

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv -d_4 - \overline{d_{4\{3c\}}} + d_3 + \overline{d_{3\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{4\{3c\}}} \cdot \overline{d_{3\{3c\}}} \pmod{3c}. \tag{8.59}$$

Then $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of $c'$. After dividing $c'$ we have

$$-84s(d_4, c) + 84s(d_3, c) \equiv -\beta + \beta a_2 a_5 \equiv 9\beta \pmod{21}.$$

because $a_2 a_5 \equiv 10 \pmod 7$ and $a_2 a_5 \equiv 1 \pmod 3$. We also have (8.54):

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}$$

because its proof does not involve whether $3|c'$ or not. Combining the two congruence equations above we can decide $\text{Arg}_2(d_3 \to d_4; \ell)$ with denominator 56 and numerator determined by $3\beta \pmod 7$ and $4 \pmod 8$, hence

$$\text{Arg}_2(d_3 \to d_4; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 6, 4, 2, 5, 3, 1.$$

This gives Table 8.26.

Comparing Tables 8.24, 8.25 and 8.26 we see that when $2|c'$, $3|c'$ and $7 \nmid c'$, Condition 8.3 holds and we have proved (8.45).

### 8.2.5 $7|c'$

This case is $49|c$ and different from the former ones. We still denote $c' = c/7$ while in this case $7|c'$, and denote $V(r, c) = \{d \pmod{c}^* : d \equiv r \pmod{c'}\}$ for $r \pmod{c'}^*$. Now $|V(r, c)| = 7$ and since $(d + c', c) = 1$

| | | | | | | |
|---|---|---|---|---|---|---|
| $c'$ (mod 42) | 6 | 12 | 18 | 24 | 30 | 36 |
| $\beta$ | 6 | 3 | 2 | 5 | 4 | 1 |
| $9\beta c'^2$ (mod 14) | 12 | 10 | 8 | 6 | 4 | 2 |
| $\text{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$ | $\frac{5}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $-\frac{1}{14}$ | $-\frac{3}{14}$ | $-\frac{5}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_3 \to d_4; 1)$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ |
| Total $\text{Arg}(d_3 \to d_4; 1)$ | $-\frac{2}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{1}{7}$ | $-\frac{1}{7}$ | $\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | $-$ | | | | | $+$ |
| $18\beta c'^2 \equiv 4c'$ (mod 7) | 3 | 6 | 2 | 5 | 1 | 4 |
| $\text{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$ | $-\frac{1}{14}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ | $\frac{3}{14}$ | $-\frac{5}{14}$ | $\frac{1}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_3 \to d_4; 2)$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ |
| Total $\text{Arg}(d_3 \to d_4; 2)$ | $\frac{2}{7}$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $\frac{3}{7}$ | $-\frac{2}{7}$ | $-\frac{2}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | | $+$ | $-$ | | |
| $81\beta c'^2$ (mod 14) | 10 | 6 | 2 | 12 | 8 | 4 |
| $\text{Arg}_1(d_3 \to d_4; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$ | $\frac{3}{14}$ | $-\frac{1}{14}$ | $-\frac{5}{14}$ | $\frac{5}{14}$ | $\frac{1}{14}$ | $-\frac{3}{14}$ |
| $(\text{Arg}_2 + \text{Arg}_3)(d_3 \to d_4; 3)$ | $\frac{5}{14}$ | $\frac{13}{14}$ | $\frac{11}{14}$ | $\frac{3}{14}$ | $\frac{1}{14}$ | $\frac{9}{14}$ |
| Total $\text{Arg}(d_3 \to d_4; 3)$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ | $\frac{3}{7}$ | $-\frac{3}{7}$ | $\frac{1}{7}$ | $\frac{3}{7}$ |
| $c'\ell \equiv \pm 1$ (mod 7)? | | $+$ | | | $-$ | |

Table 8.26: Table for $\text{Arg}(d_1 \to d_2; \ell)$ in (8.45); $2|c$, $3|c$, $7 \nmid c$.

when $(d, c) = 1$, we can write $V(r, c) = \{d, d + c', d + 2c', \cdots, d + 6c'\}$ for $1 \le d < c'$ and $d \equiv r \pmod{c'}$.

We claim that (8.1) is still true:

$$\sum_{d \in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a \ell}{7})} e\left(-\frac{12cs(d,c)}{24c}\right) e\left(\frac{5d}{c}\right) = 0, \tag{8.60}$$

while this time we have seven summation terms. The way we prove (8.60) is to show that there are only three cases for the sum: all at the outer circle (radius $1/\sin(\frac{\pi}{7})$), all at the middle circle (radius $1/\sin(\frac{2\pi}{7})$), and all at the inner circle (radius $1/\sin(\frac{3\pi}{7})$). Moreover, the seven points are equally distributed. Similar as before, we still denote $P_1$, $P_2$ and $P_3$ for each term in (8.60) and investigate the argument differences contributed from each term. Note that $P_1(d) = (-1)^{ca\ell}/\sin(\frac{\pi a \ell}{7})$ has period $c'$. Hence $\text{Arg}_1(d \to d_1; \ell) = 0$ always.

If we take any $d \in V(r, c)$ and take $a \pmod c$ such that $ad \equiv 1 \pmod c$, then for $d_* = d + c'$ and $a_* d_* \equiv 1 \pmod c$, we can take $a_* = a - c'$, $a - 2c'$, $a + 3c'$, $a + 3c'$, $a - 2c'$, $a - c'$, when $d \equiv 1, 2, 3, 4, 5, 6 \pmod 7$, respectively.

In the following two cases, we show the proof when $\ell = 1$. In the other cases $\ell = 2, 3$, only $P_1$ is affected and we still get (8.60).

#### 8.2.5.1 $c$ is odd

First we suppose $3 \nmid c$. When $d \equiv 1, 6 \pmod c$, by (8.10) we have $-12cs(d_*, c) + 12cs(d, c) \equiv 0 \pmod 6$,

$$-12cs(d_*, c) + 12cs(d, c) \equiv -d_* - a_* + d + a \equiv 0 \pmod c, \tag{8.61}$$

and

$$-12cs(d_*,c)+12cs(d,c)\equiv 2(\tfrac{d_*}{c})-2(\tfrac{d}{c})\equiv 0 \pmod 8.$$

Then $-12cs(d_*,c)+12cs(d,c)\equiv 0 \pmod{24c}$ and $\mathrm{Arg}_2(d\to d_*;\ell)=0$. Since $\mathrm{Arg}_3(d\to d_*;\ell)=\frac{5}{7}$, we have proved that the seven summands in (8.60) are equally distributed with the same radius.

When $d\equiv 2,5 \pmod 7$, only (8.61) is affected and becomes

$$-12cs(d_*,c)+12cs(d,c)\equiv -d_*-a_*+d+a\equiv c' \pmod c. \tag{8.62}$$

After dividing $24c'$ we get $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{5}{7}$ in this case and the seven points in (8.60) are still equally distributed with the same radius.

When $d\equiv 3,4 \pmod 7$, (8.61) becomes

$$-12cs(d_*,c)+12cs(d,c)\equiv -d_*-a_*+d+a\equiv -4c' \pmod c. \tag{8.63}$$

We get $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{1}{7}$ and the same conclusion as before.

Then we investigate the case $3|c'$. The following congruence

$$-12cs(d_*,c)+12cs(d,c)\equiv 2(\tfrac{d_*}{c})-2(\tfrac{d}{c})\equiv 0 \pmod 8$$

still holds and we compute

$$-12cs(d_*,c)+12cs(d,c)\equiv -d_*-\overline{d_{1\{3c\}}}+d+\overline{d_{\{3c\}}}\equiv -c'+c'\overline{d_{1\{3c\}}}\cdot\overline{d_{\{3c\}}} \pmod{3c},$$

so

$$-84s(d_*,c)+84s(d,c)\equiv -1+a_*a \pmod{21}.$$

Since $a_*a\equiv 1 \pmod 3$ and $a_*\equiv a \pmod 7$, we have

$$-84s(d_*,c)+84s(d,c)\equiv\begin{cases}0 \pmod{21} & \text{if } d\equiv 1,6 \pmod 7,\\ 9 \pmod{21} & \text{if } d\equiv 2,5 \pmod 7,\\ 15 \pmod{21} & \text{if } d\equiv 3,4 \pmod 7.\end{cases} \tag{8.64}$$

Then $-28s(d_*,c)+28s(d,c)\equiv 0,3,5 \pmod 7$ and $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{0,3,5}{7}$ by $\overline{8_{\{7\}}}\equiv 1 \pmod 7$, respectively. We still get the conclusion on equal distribution.

#### 8.2.5.2 $c$ is even

The first case is $3\nmid c'$. Congruences (8.61), (8.62) and (8.63) are still valid here. By (8.11), we define $\lambda\geq 1$ by $2^\lambda\|c'$ and claim that

$$-12cs(d_*,c)+12cs(d,c)\equiv 0 \pmod{8\times 2^\lambda} \tag{8.65}$$

To compute this, we have

$$\begin{aligned}-12cs(d_*,c)+12cs(d,c)\equiv&-d_*-\overline{d_{1\{8\times 2^\lambda\}}}(c^2+3c+1)-\overline{d_{1\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d_*})\\&+d+\overline{d_{\{8\times 2^\lambda\}}}(c^2+3c+1)+\overline{d_{\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d}) \pmod{8\times 2^\lambda}\\\equiv&-c'+c'(c^2+3c+1)\overline{d_{1\{8\times 2^\lambda\}}}\cdot\overline{d_{\{8\times 2^\lambda\}}}\\&-\overline{d_{1\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d_*})+\overline{d_{\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d}) \pmod{8\times 2^\lambda}.\end{aligned}$$

After dividing $c'$ we have

$$\begin{aligned}-84s(d_*,c)+84s(d,c)&\equiv -1+d_*d(c^2+3c+1)+2(\tfrac{c}{d_*})d_*-2(\tfrac{c}{d})d \pmod 8\\&\equiv c'(c'+1)(d+1)+2(\tfrac{c}{d_*})d_*-2(\tfrac{c}{d})d \pmod 8.\end{aligned}$$

By $2|(d+1)$, we get

$$c'(c'+1)(d+1)\equiv\begin{cases}4 \pmod 8 & \text{if } 2\|c,\ d\equiv 1 \pmod 4,\\ 0 \pmod 8 & \text{if } 2\|c,\ d\equiv 3 \pmod 4,\\ 0 \pmod 8 & \text{if } 4|c.\end{cases} \tag{8.66}$$

For $2(\frac{c}{d_*})d_*-2(\frac{c}{d})d \pmod 8$, when $\lambda\ge 2$ is even, we have $(\frac{2^\lambda}{d_*})=(\frac{2^\lambda}{d})=1$; when $\lambda\ge 3$ is odd, we have $(\frac{2}{d_*})=(\frac{2}{d})=1$. In either case, $\frac{d_*-1}{2}$ and $\frac{d-1}{2}$ are of the same parity. Hence when $4|c$, we have $(\frac{c}{d_*})d_*-(\frac{c}{d})d\equiv 0 \pmod 4$ and have proved (8.65) in this case.

When $2\|c$, we have Table 8.27 for val.:= $(\frac{c}{d_*})d_*-(\frac{c}{d})d \pmod 4$ using quadratic reciprocity. Combining

| $d \pmod 8$ | 1 | 3 | 5 | 7 |
|---|---|---|---|---|
| $d_* \pmod 8$ when $c'\equiv 2 \pmod 8$ | 3 | 5 | 7 | 1 |
| val. | 2 | 0 | 2 | 0 |
| $d_* \pmod 8$ when $c'\equiv 6 \pmod 8$ | 7 | 1 | 3 | 5 |
| val. | 2 | 0 | 2 | 0 |

Table 8.27: Table for val.:= $(\frac{c}{d_*})d_*-(\frac{c}{d})d \pmod 4$; $2|c$, no requirement for $(3,c)$, $7|c$.

Table 8.27 and (8.66) we obtain (8.65).

Combining (8.65) with (8.61), (8.62) and (8.63) shows that $\mathrm{Arg}_2(d\to d_*;\ell)$ is constant and proves the equal distribution property.

When $3|c$, we use (8.64) instead of (8.61), (8.62) and (8.63). This finishes the proof of (8.60)when $7|c'$.

Now we have proved claim (2) of Theorem 1.15.

## 8.3 Proof of Theorem 1.15, claim (3)

Here we prove claim (3) in Theorem 1.15: for all $1\le\ell\le 6$, $n\ge 0$, $7|c$ and $7\nmid A$, if $A\ell\equiv\pm 1 \pmod 7$ and $c=7A$, we have

$$e(\tfrac18)S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7)+2i\sqrt7 S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)=0. \tag{8.67}$$

We still denote $c'=c/7=A$ and $V(r,c):=\{d \pmod c)^*:\ d\equiv r \pmod c\}$ for $(r,c')=1$.

First we rewrite the two Kloosterman sums. As $\ell c \equiv \ell A \pmod 2$, by (7.5) and (7.23),

$$e(\tfrac{1}{8})S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7)=\sum_{d \pmod{c}^*}\frac{(-1)^{\ell A}\exp\left(-\frac{3\pi i c'a\ell^2}{7}\right)}{\sin(\frac{\pi a\ell}{7})}e^{-\pi i s(d,c)}e\left(\frac{(7n+5)d}{c}\right). \tag{8.68}$$

By (7.25), when $A\ell\equiv 1 \pmod 7$, we denote $T$ by $A\ell=7T+1$ and

$$\begin{aligned}&2i\sqrt{7}e(\tfrac{1}{8})S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)\\&=2i\sqrt{7}(-1)^{A\ell-[A\ell]}\sum_{B \pmod{A}^*}e\left(\frac{(\frac{3}{2}T^2+\frac{1}{2}T)C}{A}\right)e^{-\pi i s(B,A)}e\left(\frac{(7n+5)B}{A}\right);\end{aligned} \tag{8.69}$$

when $A\ell\equiv -1 \pmod 7$, we denote $T$ by $A\ell=7T-1$ (hence $A\ell-[A\ell]=A\ell-6=7(T-1)$) and

$$\begin{aligned}&2i\sqrt{7}e(\tfrac{1}{8})S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)\\&=2i\sqrt{7}(-1)^{A\ell-[A\ell]}\sum_{B \pmod{A}^*}e\left(\frac{(\frac{3}{2}(T-1)^2+\frac{5}{2}(T-1)+1)C}{A}\right)e^{-\pi i s(B,A)}e\left(\frac{(7n+5)B}{A}\right);\end{aligned} \tag{8.70}$$

For $(r,c')=1$ and any $d\in V(r,c)$, we define $P(d)$ as

$$P(d):=\frac{(-1)^{[A\ell]}e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})}e^{-\pi i s(d,c)}e\left(\frac{(7n+5)d}{c}\right)=:P_1(d)\cdot P_2(d)\cdot P_3(d). \tag{8.71}$$

When $A\ell=7T+1$, we denote $Q_1(B)=i$, $Q_3(B)=e(\frac{(7n+5)B}{A})$,

$$Q_2(B):=e\left(\frac{(\frac{3}{2}T^2+\frac{1}{2}T)C}{A}\right)e^{-\pi i s(B,A)}\quad\text{and}\quad Q(B)=:2\sqrt{7}\cdot Q_1(B)Q_2(B)Q_3(B); \tag{8.72}$$

when $A\ell\equiv -1 \pmod 7$, we only change the definition of $Q_2(B)$ to

$$Q_2(B):=e\left(\frac{(\frac{3}{2}(T-1)^2+\frac{5}{2}(T-1)+1)C}{A}\right)e^{-\pi i s(B,A)} \tag{8.73}$$

and still denote $Q(B)=2\sqrt{7}\cdot Q_1(B)Q_2(B)Q_3(B)$.

We divide the cases into subsections, which depend on $c'\ell\equiv\pm1 \pmod 7$, $\ell$, that $A$ is even or odd, and that $A$ is divisible by 3 or not. For each $r \pmod{A}^*$, recall that $d_1\in V(r,c)$ refer to the unique $d_1 \pmod{c}^*$ such that $d_1\equiv 1 \pmod 7$. We compare $P(d_1)$ and $Q(B)$ given $B=-d_1T$ and $C=-7\overline{d_{1\{A\}}}$. For the case when $c'\ell\equiv -1 \pmod 7$, we choose $c'\ell=A\ell=7T-1$, $B=d_1T$ and $C=-7\overline{d_{1\{A\}}}$. We will not repeat the proof in this case but just list a few key intermediate steps at the end.

To compare $P(d_1)$ and $Q(B)$, we denote $\mathrm{Arg}(Q_j\to P_j;\ell)$ in the following way: suppose $P_j(d_1)=Re^{i\Theta}$ and $Q_j(B)=R_Be^{i\Theta_B}$, then

$$\mathrm{Arg}(Q_j\to P_j;\ell)=\alpha\quad\text{if and only if}\quad \Theta-\Theta_B=\alpha\cdot 2\pi+2k\pi \text{ for } k\in\mathbb{Z}.$$

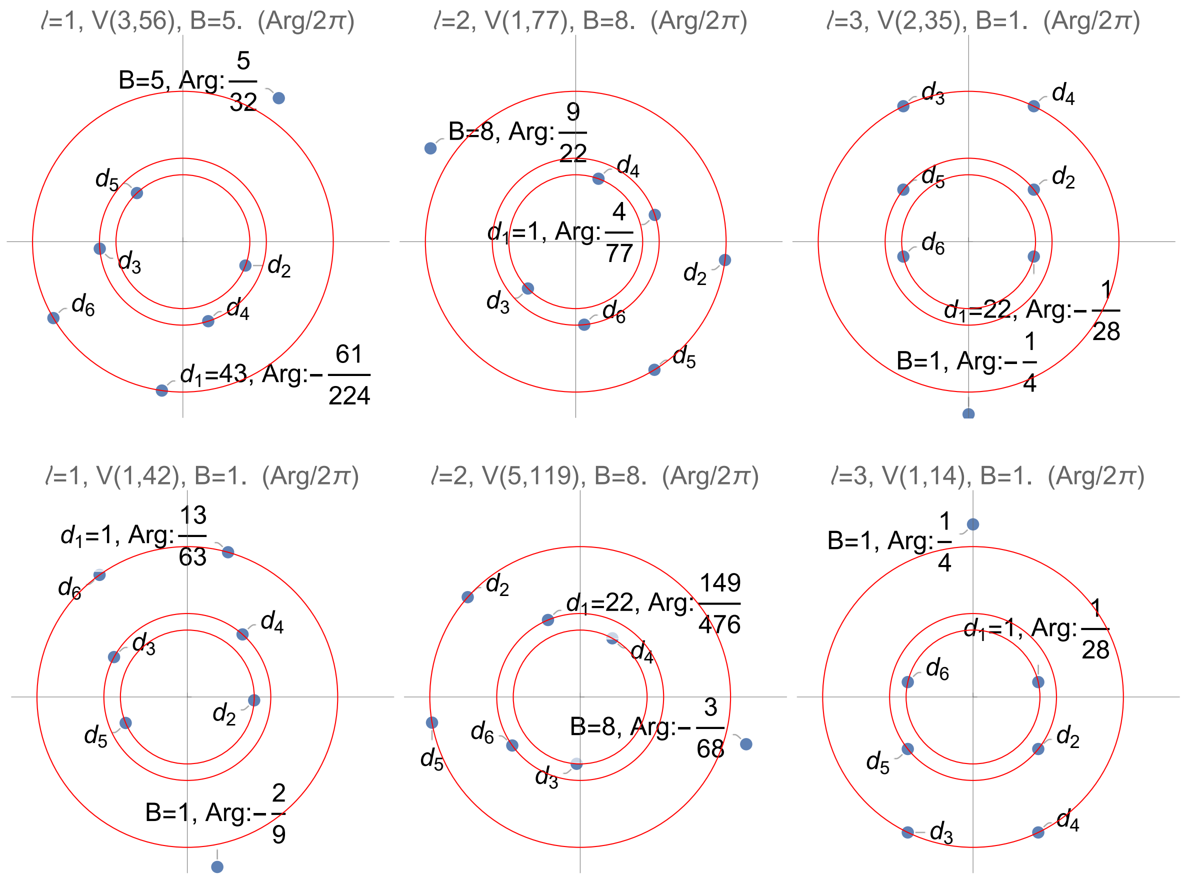
We also denote $\mathrm{Arg}(Q\to P;\ell)=\sum_{j=1}^3\mathrm{Arg}(Q_j\to P_j;\ell)$. Note that if $\mathrm{Arg}(Q_j\to P_j;\ell)=\alpha$, then $\mathrm{Arg}(Q_j\to P_j;\ell)=\alpha+k$ for all $k\in\mathbb{Z}$.

With the notations above, we claim that the argument differences have the following cases:

$$A\ell = 7T+1: \quad \mathrm{Arg}(Q \to P;\ell) = -\frac{3}{7}, -\frac{5}{14}, \frac{3}{14} \quad \text{for } \ell = 1,2,3; \tag{8.74}$$

$$A\ell = 7T-1: \quad \mathrm{Arg}(Q \to P;\ell) = \frac{3}{7}, \frac{5}{14}, -\frac{3}{14} \quad \text{for } \ell = 1,2,3. \tag{8.75}$$

To visualize the argument differences, here are a few examples:




The red circles among the figures are centered at the origin with radii $\csc(\frac{\pi}{7})$, $\csc(\frac{2\pi}{7})$, and $\csc(\frac{3\pi}{7})$, respectively, from the outside to the inside. For the styles of the six points $P(d_j)$ for $d_j \in V(r,c)$, we have the following condition. This condition has been proved by the tables in the former section, corresponding to the rows marked with "$c'\ell \equiv \pm 1 \pmod 7$?" whose entries are $+$ or $-$.

**Condition 8.4.** *When $c'\ell \equiv \pm 1 \pmod 7$, we have the following six styles for these six points $P(d)$ for $d \in V(r,c)$.*

- $\ell = 1$. *When $c'\ell \equiv 1 \pmod 7$, the arguments going $d_1 \to d_2 \to d_3 \to d_4 \to d_5 \to d_6 \to d_1$ are $\frac{3}{14}$, $-\frac{3}{7}$, $\frac{2}{7}$, $-\frac{3}{7}$, $\frac{3}{14}$, $\frac{1}{7}$, respectively. When $c'\ell \equiv -1 \pmod 7$, the direction is reversed, as shown in the second line.*

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 1 \pmod 7$ | | $\frac{3}{14}$ | | $-\frac{3}{7}$ | | $\frac{2}{7}$ | | $-\frac{3}{7}$ | | $\frac{3}{14}$ | | $\frac{1}{7}$ | |
| $c' \equiv 6 \pmod 7$ | | $-\frac{3}{14}$ | | $\frac{3}{7}$ | | $-\frac{2}{7}$ | | $\frac{3}{7}$ | | $-\frac{3}{14}$ | | $-\frac{1}{7}$ | |

- $\ell = 2$. *The first line is for* $c'\ell \equiv 1 \pmod 7$ *and the second line is for* $c'\ell \equiv -1 \pmod 7$.

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 4 \pmod 7$ | | $-\frac{1}{14}$ | | $-\frac{5}{14}$ | | $-\frac{3}{7}$ | | $-\frac{5}{14}$ | | $-\frac{1}{14}$ | | $\frac{2}{7}$ | |
| $c' \equiv 3 \pmod 7$ | | $\frac{1}{14}$ | | $\frac{5}{14}$ | | $\frac{3}{7}$ | | $\frac{5}{14}$ | | $\frac{1}{14}$ | | $-\frac{2}{7}$ | |

- $\ell = 3$. *The first line is for* $c'\ell \equiv 1 \pmod 7$ *and the second line is for* $c'\ell \equiv -1 \pmod 7$.

| | $d_1$ | $\to$ | $d_2$ | $\to$ | $d_3$ | $\to$ | $d_4$ | $\to$ | $d_5$ | $\to$ | $d_6$ | $\to$ | $d_1$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c' \equiv 5 \pmod 7$ | | $\frac{1}{7}$ | | $\frac{3}{14}$ | | $-\frac{1}{7}$ | | $\frac{3}{14}$ | | $\frac{1}{7}$ | | $\frac{3}{7}$ | |
| $c' \equiv 2 \pmod 7$ | | $-\frac{1}{7}$ | | $-\frac{3}{14}$ | | $\frac{1}{7}$ | | $-\frac{3}{14}$ | | $-\frac{1}{7}$ | | $-\frac{3}{7}$ | |

Through some simple geometry arguments, one can show that, if the six points $P(d)$ for $d \in V(r,c)$ satisfy Condition 8.4, and $\mathrm{Arg}(Q \to P; \ell)$ satisfies (8.74) and (8.75) in the corresponding cases, then we have

$$\sum_{d \in V(r,c)} P(d) + 2Q(B) = 0.$$

One hint is by using

$$\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})} + \frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})} - \frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})} = \sqrt{7}, \quad \text{where } \frac{1}{\sin(\frac{j\pi}{7})} \text{ for } j = 1,2,3 \text{ are the radii.}$$

This proves (8.67) by our choices $c = 7A$, $(A,7) = 1$, as well as the fact that $B = \mp d_1 T$ runs over all residue classes modulo $A$ and coprime to $A$ when $r$ rums over all residue classes modulo $c'$ and coprime to $c'$, for $A\ell = c'\ell \equiv \pm 1 \pmod 7$. This prove Theorem 1.15.

Subsections §8.3.1-§8.3.4 are devoted to prove (8.74), i.e. the cases $A\ell = c'\ell \equiv 1 \pmod 7$.

### 8.3.1 $c'\ell \equiv 1 \pmod 7$, $2 \nmid A$, $3 \nmid A$

Recall $d_1 \equiv 1 \pmod 7$ and $d_1 \equiv r \pmod{c'}$. Recall that we define $1 \le \beta \le 6$ as $\beta c' \equiv 1 \pmod 7$ and here $\beta = \ell$. Note that $d_1 - \beta A \equiv 7B \pmod{7A}$:

$$7B = d_1(1 - A\ell) \equiv d_1 + (7 - d_1)\ell A \pmod{7A}, \text{ so } 7B \equiv \begin{cases} 0 \pmod 7, \\ r \pmod A. \end{cases}$$

On the other hand, $d_1 - \beta c' \equiv r \pmod A$ and $d_1 - \beta c' \equiv 0 \pmod 7$. The argument difference between $P_3$ and $Q_3$ is easy to compute:

$$7\,\mathrm{Arg}(Q_3 \to P_3; \ell) \equiv 5 d_1 \ell \equiv 5\ell \pmod 7 \tag{8.76}$$

which does not depend on $n$.

Recall $\overline{d_{1\{7A\}}} \equiv a_1 \pmod{7A}$ and $\overline{B_{\{A\}}} \equiv 7\overline{d_{1\{A\}}} \pmod A$. We have

$$\begin{aligned} -84A(s(d_1,7A) - s(B,A)) &\equiv -d_1 - a_1 + d_1(1 - \beta A) + 49\overline{d_{1\{A\}}} \\ &\equiv -d_1\beta A - a_1 + 49\overline{d_{1\{A\}}} \pmod{7A}. \end{aligned}$$

Hence

$$-84A(s(d_1,7A) - s(B,A)) \equiv \begin{cases} -2 \pmod 7 \\ 48\overline{d_{1\{A\}}} \pmod A \end{cases} \tag{8.77}$$

We also have

$$\begin{aligned}-84A(s(d_1,7A)-s(B,A)) &\equiv -7A-1+2(\tfrac{d_1}{7A})+7(A+1)-14(\tfrac{B}{A})\\ &\equiv 6+2(\tfrac{d_1}{A})+2(\tfrac{d_1}{A})(\tfrac{7}{A}) \pmod 8,\end{aligned}$$

where the last step is because $(A,7)=1$, $(\frac{d_1}{7})=(\frac{1}{7})=1$ and $7B\equiv d_1 \pmod A$. By $A$ is odd and $A\ell\equiv 1 \pmod T$, we have $(\frac{7}{A})=(\frac{\ell}{7})(-1)^{\frac{A-1}{2}}$. Combining $6|12cs(d_1,c)$ and $6|12As(B,A)$ we conclude

$$-84A(s(d_1,7A)-s(B,A))\equiv\begin{cases}18 \pmod{24}, & \text{if } \quad A\equiv 1 \pmod 4 \text{ which requires:}\\ & \qquad \ell=1,\ 4|T;\\ & \qquad \ell=2,\ T\equiv 7 \pmod 8;\\ & \text{or if } \quad A\equiv 3 \pmod 4 \text{ which requires:}\\ & \qquad \ell=3,\ T\equiv 8 \pmod{12};\\ \\ 6 \pmod{24}, & \text{if } \quad A\equiv 3 \pmod 4 \text{ which requires:}\\ & \qquad \ell=1,\ 2\|T;\\ & \qquad \ell=2,\ T\equiv 3 \pmod 8;\\ & \text{or if } \quad A\equiv 1 \pmod 4 \text{ which requires:}\\ & \qquad \ell=3,\ T\equiv 2 \pmod{12}.\end{cases} \tag{8.78}$$

Next we check the part of $Q_2$ other than $e^{-\pi i s(B,A)}$. Since $A$ is odd and $T$ is even, we have

$$\begin{aligned}\left(\tfrac{3}{2}T^2+\tfrac{1}{2}T\right)C &\equiv \tfrac{T}{2}(3T+1)(-7\overline{d_{1\{A\}}})\\ &\equiv \tfrac{T}{2}(3-3A\ell-7)\overline{d_{1\{A\}}}\\ &\equiv -2T\overline{d_{1\{A\}}} \pmod A.\end{aligned}$$

Then the part of $Q_2$ other than $e^{-\pi i s(d,c)}$ is

$$e\left(\frac{24\cdot 2\overline{d_{1\{A\}}}(-7T)}{24\cdot 7A}\right)=e\left(\frac{48\overline{d_{1\{A\}}}(1-A\ell)}{168A}\right),\ \text{with numerator}\ \equiv\begin{cases}0 \pmod 7,\\ 48\overline{d_{1\{A\}}} \pmod A,\\ 0 \pmod{24}.\end{cases} \tag{8.79}$$

We conclude that

$$24\cdot 7A\,\mathrm{Arg}(Q_2\to P_2;\ell)\equiv R_2 \pmod{168A} \tag{8.80}$$

where $R_2$ is determined by (8.77), (8.78) and (8.79): $R_2\equiv 0 \pmod A$, $R_2\equiv -2 \pmod 7$, and $R_2\equiv 18,6 \pmod{24}$ depending on the cases in (8.78). Therefore, by $A\ell\equiv 1 \pmod 7$ and $A \pmod 4$ in (8.78) we conclude

$$\mathrm{Arg}(Q_2\to P_2;\ell)=\frac{23,\ 11,\ 13}{28} \quad \text{for } \ell=1,2,3. \tag{8.81}$$

Then we compute $\mathrm{Arg}(Q_1\to P_1;\ell)$. When $\ell=1$, since $A$ is odd, $A\equiv 1 \pmod{14}$. Note that both $a_1\equiv 1,8 \pmod{14}$ give the same result due to the sign of $\sin(\frac{\pi a}{7})$. It is direct to get (remember $Q_1=i$)

$$\mathrm{Arg}(Q_1\to P_1;1)=\frac{1}{2}-\frac{3}{14}-\frac{1}{4}=\frac{1}{28}. \tag{8.82}$$

When $\ell = 2$, we get $A \equiv 4 \pmod 7$ and

$$\mathrm{Arg}(Q_1 \to P_1; 2) = \frac{1}{2} - \frac{3}{7} - \frac{1}{4} = -\frac{5}{28}. \tag{8.83}$$

When $\ell = 3$, we have $A \equiv 5 \pmod{14}$ and both $a_1 \equiv 1, 8 \pmod{14}$ gives the same result. We get

$$\mathrm{Arg}(Q_1 \to P_1; 3) = \frac{1}{2} - \frac{9}{14} - \frac{1}{4} = -\frac{11}{28}. \tag{8.84}$$

Combining (8.82), (8.83), (8.84), (8.81), and (8.76), we get

$$\mathrm{Arg}(Q \to P; \ell) = -\frac{3}{7},\ -\frac{5}{14},\ \frac{3}{14} \quad \text{for } \ell = 1, 2, 3. \tag{8.85}$$

This proves the claim (8.74).

### 8.3.2 $c'\ell \equiv 1 \pmod 7$, $2 \nmid A$, $3|A$

In this case (8.76) still holds. For $\mathrm{Arg}(Q_2 \to P_2; \ell)$, by (8.9) we have

$$-84A(s(d_1, 7A) - s(B, A)) \equiv -d_1 A\ell - \overline{d_{1\{21A\}}} + 7\overline{(-d_1 T)_{\{3A\}}} \pmod{21A}.$$

We have

$$\begin{aligned} -84A(s(d_1, 7A) - s(B, A)) &\equiv -d_1 A\ell - \overline{d_{1\{3A\}}} + 49\overline{(d_1 - d_1 A\ell)_{\{3A\}}} \\ &\equiv -d_1 A\ell + (48d_1 + d_1 A\ell)\overline{d_{1\{3A\}}}\,\overline{(d_1 - d_1 A\ell)_{\{3A\}}} \\ &\equiv d_1 A\ell \left(\overline{d_{1\{3A\}}}\,\overline{(d_1 - d_1 A\ell)_{\{3A\}}} - 1\right) + 48\overline{d_{1\{A\}}} \\ &\equiv 48\overline{d_{1\{A\}}} \pmod{3A} \end{aligned} \tag{8.86}$$

where in the second congruence we used

$$\overline{(x+y)_m} - 49\overline{x_{\{m\}}} \equiv \overline{x_{\{m\}}}\,\overline{(x+y)_{\{m\}}}(-48x - 49y) \pmod m$$

for $(x + y, m) = (x, m) = 1$ and in the last two congruences we used

$$m_1\overline{x_{\{m_1 m_2\}}} \equiv m_1\overline{x_{\{m_2\}}} \pmod{m_1 m_2} \tag{8.87}$$

for $(x, m_1 m_2) = 1$. We still have

$$-\,84A(s(d_1, 7A) - s(B, A)) \equiv -2 \pmod 7. \tag{8.88}$$

Moreover, (8.78) and (8.79) still hold except the second congruence in (8.79) should be changed to $48\overline{d_{1\{A\}}} \pmod{3A}$.

We conclude

$$24 \cdot 7A\,\mathrm{Arg}(Q_2 \to P_2; \ell) \equiv R_2 \pmod{168A} \tag{8.89}$$

where $R_2$ is determined by (8.86), (8.88), (8.78) and (8.79): $R_2 \equiv 0 \pmod{3A}$, $R_2 \equiv -2 \pmod 7$, and $R_2 \equiv 18, 6 \pmod{24}$ depending on the cases in (8.78). Therefore, by $A\ell \equiv 1 \pmod 7$ and $A \pmod 4$ in (8.78) we conclude

$$\operatorname{Arg}(Q_2 \to P_2; \ell) = \frac{23,\ 11,\ 13}{28} \quad \text{for } \ell = 1, 2, 3. \tag{8.90}$$

The condition $3|A$ does not affect $\operatorname{Arg}(Q_1 \to P_1; \ell)$ and $\operatorname{Arg}(Q_3 \to P_3; \ell)$. Combining (8.90) with (8.82), (8.83), (8.84), and (8.76), we have proved (8.74) in this case.

### 8.3.3 $c'\ell \equiv 1 \pmod 7$, $2|A$, $3 \nmid A$

Recall (8.76) which will be unchanged, while for $\operatorname{Arg}(Q_2 \to P_2; \ell)$ we have (8.77) and need to use (8.11). Let $\lambda \geq 1$ be defined as $2^\lambda \| A$. Recall $B = -d_1 T$ and $7T + 1 = A\ell$. We have

$$\begin{aligned}
&-84A(s(d_1, 7A) - s(B, A)) \\
&\quad \equiv -d_1 - \overline{d_1}_{\{8\times 2^\lambda\}}(49A^2 + 21A + 1) - 14\overline{d_1}_{\{8\times 2^\lambda\}}A(\tfrac{7A}{d_1}) \\
&\qquad + d_1(1 - A\ell) + 49\overline{(d_1 - d_1A\ell)}_{\{8\times 2^\lambda\}}(A^2 + 3A + 1) + 14\overline{B}_{\{8\times 2^\lambda\}}A(\tfrac{A}{B}) \\
&\quad \equiv -d_1A\ell + 49A^2 \cdot d_1A\ell\overline{(d_1 - d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}} \\
&\qquad + 21A(6d_1 + d_1A\ell)\overline{(d_1 - d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}} \\
&\qquad + (48d_1 + d_1A\ell)\overline{(d_1 - d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}} \\
&\qquad + 14A\left(\overline{B}_{\{8\times 2^\lambda\}}(\tfrac{A}{B}) - \overline{d_1}_{\{8\times 2^\lambda\}}(\tfrac{7A}{d_1})\right) \pmod{8 \times 2^\lambda}.
\end{aligned}$$

Since $2^\lambda \| A$ with $\lambda \geq 1$, we apply (8.87) and $x^2 \equiv 1 \pmod 8$ (for odd $x$) to get

$$\begin{aligned}
-84A(s(d_1, 7A) - s(B, A)) &\equiv 6d_1A + d_1A^2\ell(1 + \ell) + 48\overline{d_1}_{\{A\}} \\
&\quad + 6A\left(B(\tfrac{A}{B}) - d_1(\tfrac{7A}{d_1})\right) \pmod{8 \times 2^\lambda}.
\end{aligned}$$

By (8.88), To determine $B(\frac{A}{B}) - d_1(\frac{7A}{d_1}) \pmod 4$, we use the quadratic reciprocity

$$(\tfrac{x}{y})(\tfrac{y}{x}) = \pm(-1)^{\frac{x_o - 1}{2}\frac{y_o - 1}{2}}, \quad \text{where } x_o \text{ is the odd part of } x$$

and the $\pm$ sign is $+$ if $x \geq 0$ or $y \geq 0$ and is $-$ if $x < 0$ and $y < 0$. By $B < 0$ odd and $A > 0$, we compute

$$\begin{aligned}
B(\tfrac{A}{B}) - d_1(\tfrac{7A}{d_1}) &\equiv -d_1T(\tfrac{B}{A})(-1)^{\frac{\frac{A}{2^\lambda} - 1}{2}\cdot\frac{B-1}{2}} - d_1(\tfrac{d_1}{7A})(-1)^{\frac{7\cdot\frac{A}{2^\lambda} - 1}{2}\cdot\frac{d_1 - 1}{2}} \\
&\equiv -d_1T(\tfrac{d_1 - d_1A\ell}{A})(\tfrac{7}{A})(-1)^{\frac{\frac{A}{2^\lambda} - 1}{2}\cdot\frac{B-1}{2}} - d_1(\tfrac{d_1}{A})(-1)^{\frac{7\cdot\frac{A}{2^\lambda} - 1}{2}\cdot\frac{d_1 - 1}{2}} \pmod 4
\end{aligned} \tag{8.91}$$

Here are the cases:

1. If $4|A$, then we have $T \equiv 1 \pmod 4$, $B \equiv -d_1 \pmod 4$. Moreover, $(\frac{d_1 - d_1A\ell}{A}) = (\frac{d_1}{A})$ always (note that $A$ is even and we have to consider $(\frac{d_1}{2})$). Now the above congruence (8.91) simplifies to $(\frac{\ell}{7})d_1 + 1 \pmod 4$. In this case $d_1A^2\ell(1 + \ell) \equiv 0 \pmod{8 \times 2^\lambda}$ and we conclude

$$-84A(s(d_1, 7A) - s(B, A)) \equiv \begin{cases} 2A + 48\overline{d_1}_{\{A\}} \pmod{8 \times 2^\lambda}, & \ell = 1, 2; \\ 6A + 48\overline{d_1}_{\{A\}} \pmod{8 \times 2^\lambda}, & \ell = 3. \end{cases} \tag{8.92}$$

2. If $2\|A$ and $\ell = 1$, then $T \equiv 3 \pmod 4$, $B \equiv d_1 \pmod 4$ and the above (8.91) simplifies to $d_1 - 1 \pmod 4$. Then as $A(12d_1 - 6 + 2d_1A) \equiv 2A \pmod{8 \times 2^\lambda}$, we conclude the same as the first line of (8.92).

3. If $2\|A$ and $\ell=2$, then $T\equiv 1 \pmod 4$, $B\equiv -d_1 \pmod 4$, and $(\frac{d_1-d_1A\ell}{A})=-(\frac{d_1}{A})$. Now (8.91) gives $d_1-1 \pmod 4$ and we again get the first line of (8.92).

4. If $2\|A$ and $\ell=3$, then $T\equiv 3 \pmod 4$, $B\equiv d_1 \pmod 4$, and $(\frac{A}{7})=(\frac{3A}{7})(\frac{3}{7})=-1$. Here (8.91) results in $d_1-1 \pmod 4$ again. Note that $d_1A^2\ell(1+\ell)\equiv 0 \pmod{8\times 2^\lambda}$ and we get the second line of (8.92).

Next we check the part of $Q_2$ other than $e^{-\pi i s(d,c)}$. In this case $A$ is even, so $3T+1$ is even and we have

$$\left(\tfrac{3}{2}T^2+\tfrac{1}{2}T\right)C\equiv \tfrac{3T+1}{2}\cdot T(-7\overline{d_{1\{A\}}})\equiv \frac{3T+1}{2}\overline{d_{1\{A\}}} \pmod A.$$

When written with denominator $24\cdot 7A$, we have

$$e\left(\frac{(\frac{3}{2}T^2+\frac{1}{2}T)C}{A}\right)=e\left(\frac{36A\ell\overline{d_{1\{A\}}}+48\overline{d_{1\{A\}}}}{24\cdot 7A}\right)$$

whose numerator is

$$36A\ell\overline{d_{1\{A\}}}+48\overline{d_{1\{A\}}}\equiv\begin{cases}0 \pmod 7,\\ 48\overline{d_{1\{A\}}} \pmod{3A},\\ 4A+48\overline{d_{1\{A\}}} \pmod{8\times 2^\lambda}, & \ell=1,3,\\ 48\overline{d_{1\{A\}}} \pmod{8\times 2^\lambda}, & \ell=2.\end{cases} \tag{8.93}$$

Combining the above computation with (8.77), (8.92) and (8.8), we get

$$\operatorname{Arg}(Q_2\to P_2;\ell)=\frac{9,11,27}{28}\quad \text{for } \ell=1,2,3. \tag{8.94}$$

Then we compute $\operatorname{Arg}(Q_1\to P_1;\ell)$. When $\ell=1$, since $A$ is even, $\frac{A}{2}\equiv 4 \pmod 7$. Note that $a_1\equiv 1 \pmod{14}$ because $a_1$ is odd. It is direct to get (remember $Q_1=i$)

$$\operatorname{Arg}(Q_1\to P_1;1)=\frac{1}{2}-\frac{5}{7}-\frac{1}{4}=-\frac{13}{28}. \tag{8.95}$$

When $\ell=2$, we get $\frac{A}{2}\equiv 2 \pmod 7$ and

$$\operatorname{Arg}(Q_1\to P_1;2)=\frac{1}{2}-\frac{3}{7}-\frac{1}{4}=-\frac{5}{28}. \tag{8.96}$$

When $\ell=3$, we have $\frac{A}{2}\equiv 6 \pmod{14}$ and

$$\operatorname{Arg}(Q_1\to P_1;3)=\frac{1}{2}-\frac{1}{7}-\frac{1}{4}=\frac{3}{28}. \tag{8.97}$$

Combining (8.95), (8.96), (8.97), (8.94), and (8.76), we get

$$\operatorname{Arg}(Q\to P;\ell)=-\frac{3}{7},-\frac{5}{14},\frac{3}{14}\quad \text{for } \ell=1,2,3. \tag{8.98}$$

### 8.3.4 $c'\ell\equiv 1 \pmod 7$, $2|A$, $3|A$

Comparing to the former case, the only difference in getting $\operatorname{Arg}(Q_2\to P_2;\ell)$ in (8.94) is that we should using (8.86) instead of (8.77). The result (8.94) still holds in this case. The condition $3|A$ or $3\nmid A$ does not affect the computation for $\mathrm{Arg}(Q_1 \to P_1; \ell)$ and $\mathrm{Arg}(Q_3 \to P_3; \ell)$, hence we still have

$$\mathrm{Arg}(Q \to P; \ell) = -\frac{3}{7}, -\frac{5}{14}, \frac{3}{14} \quad \text{for } \ell = 1, 2, 3. \tag{8.99}$$

Now we have finished the discussion in all the cases for $A$ when $A\ell \equiv 1 \pmod 7$ and proved (8.67). For the other case $A\ell \equiv -1 \pmod 7$, we will not repeat the same process but just list the key argument differences below. For every $r \pmod{c'}^*$, we compare $P(d)$ (8.71) given $d = d_1 \in V(r, c)$ and $Q(B)$ (8.73) given $T := \frac{A\ell+1}{7} > 0$, $B = d_1 T$ and $C = -7\overline{d_{1\{A\}}}$. Now $7B = d_1 + d_1 A\ell$. We shall get Table 8.28.

| Case $2 \nmid A$: | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ |
|---|---|---|---|
| $\mathrm{Arg}(Q_1 \to P_1; \ell)$ | $-\frac{1}{28}$ | $\frac{5}{28}$ | $\frac{11}{28}$ |
| $\mathrm{Arg}(Q_2 \to P_2; \ell)$ | $\frac{5}{28}$ | $-\frac{11}{28}$ | $-\frac{13}{28}$ |
| $\mathrm{Arg}(Q_3 \to P_3; \ell)$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ |
| $\mathrm{Arg}(Q \to P; \ell)$ | $\frac{3}{7}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ |
| Case $2 \mid A$: | $\ell = 1$ | $\ell = 2$ | $\ell = 3$ |
| $\mathrm{Arg}(Q_1 \to P_1; \ell)$ | $\frac{13}{28}$ | $\frac{5}{28}$ | $-\frac{3}{28}$ |
| $\mathrm{Arg}(Q_2 \to P_2; \ell)$ | $-\frac{9}{28}$ | $-\frac{11}{28}$ | $\frac{1}{28}$ |
| $\mathrm{Arg}(Q_3 \to P_3; \ell)$ | $\frac{2}{7}$ | $-\frac{3}{7}$ | $-\frac{1}{7}$ |
| $\mathrm{Arg}(Q \to P; \ell)$ | $\frac{3}{7}$ | $\frac{5}{14}$ | $-\frac{3}{14}$ |

Table 8.28: Table for the case $A\ell \equiv -1 \pmod 7$

We have finished the proof of claim (3) in Theorem 1.15 and finished the proof of this theorem.

## 8.4 An extra coincidence

The following lemma may provide some information about the key equation defining $x_r$ for every integer $r \geq 0$ in (2.37):

$$\frac{3}{2}x^2 - \frac{2r+1}{2}x + \frac{1}{24} = 0.$$

**Lemma 8.5.** *For $k \in \mathbb{Z}$ and $k \geq 2$, let $x_k$ be the only root in $(0, \frac{1}{2})$ of the quadratic equation $\frac{3}{2}x^2 - \frac{2k-1}{2}x + \frac{1}{24} = 0$. Let $0 \leq [p]_{24} < 24$ denote the residue of a prime $p$ modulo 24. Then the following two quantities are equal:*

*(1) the smallest prime $p > x_k^{-1}$;*

*(2) the smallest prime $p$ such that $S_{\frac{p-2}{2}}(\mathrm{SL}_2(\mathbb{Z}), \nu_\eta^{-p}) = k$, where $S_w(\Gamma, \nu)$ is the space of weight $w$ holomorphic cusp forms on $\Gamma$ with multiplier system $\nu$.*

*Remark.* Let $\lfloor x \rfloor$ denote the floor of $x$. Since $\frac{[p]_{24}+p-2}{2} \not\equiv 2 \pmod{12}$ for every prime $p \neq 3$, we have

$$\dim S_{\frac{p-2}{2}}(\mathrm{SL}_2(\mathbb{Z}), \nu_\eta^{-p}) = \dim S_{\frac{[p]_{24}+p-2}{2}}(\mathrm{SL}_2(\mathbb{Z})) = \left\lfloor \frac{[p]_{24} + p - 2}{24} \right\rfloor.$$

Clearly the dimension of this space is non-decreasing when $p$ increases.

*Proof.* First observe that when $k = 2$, we have $x_2^{-1} = 34.97\cdots$ and $p = 37$ is the prime for both (1) and (2). Then we consider $k \geq 3$.

Let the positive integer $t$ be defined as $p = [p]_{24} + 24t$. Then

$$\left\lfloor \frac{[p]_{24} + p - 2}{24} \right\rfloor = \begin{cases} t & 1 \le [p]_{24} \le 11, \\ t+1 & 13 \le [p]_{24} \le 23. \end{cases}$$

This shows that for $p$ in condition (2) we must have $13 \le [p]_{24} \le 23$ and $k = t+1$. Hence, $p$ in condition (2) is the smallest prime such that

$$\frac{[p]_{24} + p - 2}{24} \ge k \;\Leftrightarrow\; [p]_{24} + 12t \ge 12k + 1 \;\Leftrightarrow\; p \ge 24k - 11.$$

We claim that $-2 < x_k^{-1} - (24k - 11) < 0$. If this is true, since $24k - 11$ is odd, there is no prime between $x_k^{-1}$ and $24k - 11$, hence the lemma is proved. It is easy to compute that $x_k^{-1} = 12k - 6 + 12\sqrt{k^2 - k}$. When $k \ge 3$, we have

$$\sqrt{k^2 - k} - k = \frac{-1}{\sqrt{1 - 1/k} + 1} \in \left( \frac{-1}{1 + \sqrt{2/3}},\ -\frac{1}{2} \right) \subset \left( -\frac{5}{9},\ -\frac{1}{2} \right).$$

Then

$$x_k^{-1} - (24k - 11) = 5 + 12\left(\sqrt{k^2 - k} - k\right) \in \left( -\frac{15}{9}, -1 \right) \subset (-2, 0).$$

This finishes the proof. □

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