VANISHING PROPERTIES OF KLOOSTERMAN SUMS AND DYSON'S CONJECTURES, WHOLE PROOF

QIHANG SUN

Abstract. In a previous paper [Sun24], the author proved the exact formulae for ranks of partitions modulo each prime $p \geq 5$. In this paper, for $p = 5$ and 7, we prove special vanishing properties of the Kloosterman sums appearing in the exact formulae. These vanishing properties imply a new proof of Dyson's rank conjectures. Specifically, we give a new proof of Ramanujan's congruences $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$.

1. Introduction

Let $p(n)$ denote the integer partition function. Ramanujan obtained the famous congruence properties of $p(n)$:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7, \quad p(11n+6) \equiv 0 \pmod{11}. \tag{1.1}$$

In 1944, Dyson [Dys44] defined the rank of a partition and conjectured a beautiful explanation for Ramanujan's congruences. Suppose $\Lambda = \{\Lambda_1 \geq \Lambda_2 \geq \cdots \geq \Lambda_\kappa\}$ is a partition of n, i.e. $\sum_{j=1}^{\kappa} \Lambda_j = n$. Then the rank of Λ is defined by

$$\mathrm{rank}(\Lambda) := \Lambda_1 - \kappa$$

Let the quantities $N(m,n)$ and $N(a,b;n)$ be defined by

$$N(m,n) := \#\{\Lambda \text{ is a partition of } n : \mathrm{rank}\,\Lambda = m\} \tag{1.2}$$

and

$$N(a,b;n) := \#\{\Lambda \text{ is a partition of } n : \mathrm{rank}\,\Lambda \equiv a \pmod b\}. \tag{1.3}$$

Let $q = \exp(2\pi i z) = e(z)$ for $z \in \mathbb{H}$ and w be a root of unity. The generating function of $N(m,n)$ is given by (see e.g. [BO06, p. 245])

$$\mathcal{R}(w;q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n}, \tag{1.4}$$

where $(a;q)_n := \prod_{j=0}^{n-1}(1 - aq^j)$. Dyson made the following conjectures which were proved by Atkin and Swinnerton-Dyer in 1953.

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Theorem 1.1 ([ASD54]). *For all $n \geq 0$, we have the following identities:*

$$N(1,5;5n+1) = N(2,5;5n+1); \tag{5-1}$$

$$N(0,5;5n+2) = N(2,5;5n+2); \tag{5-2}$$

$$N(0,5;5n+4) = N(1,5;5n+4) = N(2,5;5n+4); \tag{5-4}$$

$$N(2,7;7n) = N(3,7;7n); \tag{7-0}$$

$$N(1,7;7n+1) = N(2,7;7n+1) = N(3,7;7n+1); \tag{7-1}$$

$$N(0,7;7n+2) = N(3,7;7n+2); \tag{7-2}$$

$$N(0,7;7n+3) = N(2,7;7n+3), \quad N(1,7;7n+3) = N(3,7;7n+3); \tag{7-3}$$

$$N(0,7;7n+4) = N(1,7;7n+4) = N(3,7;7n+4); \tag{7-4}$$

$$N(0,7;7n+5) = N(1,7;7n+5) = N(2,7;7n+5) = N(3,7;7n+5); \tag{7-5}$$

$$N(0,7;7n+6) + N(1,7;7n+6) = N(2,7;7n+6) + N(3,7;7n+6). \tag{7-6}$$

Remark. By $N(a,b;n) = N(-a,b;n)$, the identity (5-4) implies

$$N(0,5;5n+4) = N(1,5;5n+4) = \cdots = N(4,5;5n+4) = \tfrac{1}{5}p(5n+4)$$

hence $p(5n+4) \equiv 0 \pmod 5$. The identity (7-5) implies

$$N(0,7;7n+5) = N(1,7;7n+5) = \cdots = N(6,7;7n+5) = \tfrac{1}{7}p(7n+5)$$

hence $p(7n+5) \equiv 0 \pmod 7$.

The proof of Theorem 1.1 in [ASD54] involves identities of generating functions

$$\sum_{n=0}^{\infty} \left(N(a,p;pn+k) - N(b,p;pn+k)\right) x^{pn} \prod_{r=1}^{\infty}(1-x^r)$$

for $p = 5, 7$ and certain choices of the integer k. See [ASD54, Theorem 4 & Theorem 5] for details. Recently, Garvan [Gar17, §6] gave a new and simplified proof of Dyson's conjectures. For

$$\mathcal{K}_{p,0}(z) = \prod_{n=1}^{\infty}(1-q^{pn}) \sum_{n=\lceil (p^2-1)/24p \rceil} \left(\sum_{k=0}^{p-1} N(k,p;pn - \tfrac{p^2-1}{24}) \zeta_p^k \right) q^n,$$

with $\zeta_p := e(\frac{1}{p})$ and $\mathcal{K}_{p,m}(z)$ defined in [Gar17, Definition 6.1], Garvan showed that $\mathcal{K}_{p,0}(z)$ is a weakly holomorphic modular form of weight 1 on $\Gamma_1(p)$. By the Valence formula, Garvan proved $\mathcal{K}_{5,0}(z) = \mathcal{K}_{7,0}(z) = 0$ and hence proved the Dyson's conjectures in [Gar17, §6.3].

For integers $b > a > 0$, denote $A(\frac{a}{b};n)$ as the Fourier coefficient of $\mathcal{R}(\zeta_b^a;q)$:

$$\mathcal{R}(\zeta_b^a;q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{b};n\right) q^n$$

where $\zeta_b = \exp(\frac{2\pi i}{b})$ is a b-th root of unity. There is an important equation which explains the relation between $A(\frac{a}{b};n)$ and $N(a,b;n)$:

$$bN(a,b;n) = p(n) + \sum_{j=1}^{b-1} \zeta_b^{-aj} A\left(\frac{j}{b};n\right). \tag{1.5}$$

It is not hard to show that $A(\frac{j}{b};n) \in \mathbb{R}$ and $A(\frac{j}{b};n) = A(1-\frac{j}{b};n)$ for $1 \le j \le b-1$, because $N(a,b;n) = N(-a,b;n)$ and $\zeta_b^{-aj} + \zeta_b^{-a(b-j)} = 2\cos(\frac{\pi a j}{b})$. Specifically, if we know the values of $A(\frac{j}{b};n)$ for $1 \le j \le b-1$, then we know the value of

$$N(a_1,b;n) - N(a_2,b;n) \quad \text{for any } 0 \le a_1, a_2 \le b-1.$$

Another way of approaching Dyson's conjectures is therefore via the formulae for $A(\frac{j}{b};n)$ when $b=5,7$. In 2009, Bringmann [Bri09, Theorem 1.1] proved the asymptotic formula for $A(\frac{j}{b};n)$ when $b \ge 3$ is odd and $0 \le j \le b-1$. Bringmann used the asymptotic formula when $b=3$ to prove the Andrews-Lewis conjecture about comparing $N(0,3;n)$ and $N(1,3;n)$. In a previous paper [Sun24], for each prime $p \ge 5$ and $1 \le \ell \le p-1$, the author proved that Bringmann's asymptotic formula, when summing up to infinity, is the exact formula for $A(\frac{\ell}{p};n)$ for all $n \ge 0$. For each integer A, denote $[A]$ by

$$0 \le [A] < 7: \quad [A] \equiv A \pmod 7.$$

When $p=5$ or 7, that exact formula reduces to the following corollary.

Theorem 1.2 ([Sun24, Corollary 2.2])**.** *For every positive integer n, when $p=5$ and $1 \le \ell \le 4$, we have*

$$A\left(\frac{\ell}{5};n\right) = \frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{5})}{(24n-1)^{\frac{1}{4}}} \sum_{c>0:\, 5|c} \frac{S_{\infty\infty}^{(\ell)}(0,n,c,\mu_5)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right); \tag{1.6}$$

when $p=7$ and $1 \le \ell \le 6$, we have

$$\begin{aligned} A\left(\frac{\ell}{7};n\right) = &\frac{2\pi e(-\frac{1}{8})\sin(\frac{\pi\ell}{7})}{(24n-1)^{\frac{1}{4}}} \sum_{c>0:\, 7|c} \frac{S_{\infty\infty}^{(\ell)}(0,n,c,\mu_7)}{c} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24c}\right) \\ &+ \frac{4\pi\sin(\frac{\pi\ell}{7})}{(24n-1)^{\frac{1}{4}}} \sum_{\substack{a>0:\, 7\nmid a,\\ [a\ell]=1 \text{ or } 6}} \frac{S_{0\infty}^{(\ell)}(0,n,a,\mu_7;0)}{\sqrt{7}\,a} I_{\frac{1}{2}}\left(\frac{4\pi\sqrt{24n-1}}{24\times 7a}\right). \end{aligned} \tag{1.7}$$

Here $S_{\infty\infty}^{(\ell)}(0,n,c,\mu_p)$ for $p=5,7$ and $S_{0\infty}^{(\ell)}(0,n,a,\mu_7;0)$ are given in (2.7), (2.8) *and* (2.9).

Theorem 1.2 gives us a new way to directly compute $A(\frac{\ell}{p};n)$ for $p=5,7$ and all ℓ. In this paper, we give a new proof of Theorem 1.1 by establishing the following vanishing properties of the Kloosterman sums $S_{\infty\infty}^{(\ell)}(0,n,c,\mu_p)$ for $p=5,7$ and $S_{0\infty}^{(\ell)}(0,n,a,\mu_7;0)$. This is totally different from the methods in [ASD54] and [Gar17].

Theorem 1.3. *(i) For all integers $n \ge 0$ and $1 \le \ell \le p-1$ for $p=5,7$ (denoted by $p|c$ below), we have the following vanishing conditions for the Kloosterman sums appeared in Theorem 1.2:*

(5-4) *If $5|c$, we have $S_{\infty\infty}^{(\ell)}(0,5n+4,c,\mu_5)=0$.*
(7-5,1) *If $7|c$, $\frac{c}{7}\cdot\ell \not\equiv 1 \pmod 7$, and $\frac{c}{7}\cdot\ell \not\equiv -1 \pmod 7$, then $S_{\infty\infty}^{(\ell)}(0,7n+5,c,\mu_7;0)=0$.*
(7-5,2) *If $7|c$, $7\nmid a$, $a\ell \equiv \pm 1 \pmod 7$, and $c=7a$, we have*

$$e(-\tfrac{1}{8})S_{\infty\infty}^{(\ell)}(0,7n+5,c,\mu_7) + 2\sqrt{7}\,S_{0\infty}^{(\ell)}(0,7n+5,a,\mu_7;0) = 0.$$

(ii) Furthermore, we denote $C_p^{a,b} := \cos(\frac{a\pi}{p}) - \cos(\frac{b\pi}{p})$ *and*

$$S_7^{(\ell)}(n,c) := \sin(\tfrac{\pi\ell}{7})\left(e(-\tfrac{1}{8})S_{\infty\infty}^{(\ell)}(0,n,c,\mu_7) + \mathbf{1}_{\substack{a:=c/7\\ [a\ell]=1,6}} \cdot 2\sqrt{7}\, S_{0\infty}^{(\ell)}(0,7n+5,a,\mu_7;0)\right)$$

for simplicity, where $\mathbf{1}_{condition}$ *equals 1 if the condition meets and equals 0 otherwise. We also have the following vanishing conditions for all* $c \in \mathbb{Z}$ *divisible by* p*, where* $p = 5$ *or* 7 *is marked at the subscript of* $C_p^{a,b}$*:*

$$C_5^{2,4}\sin(\tfrac{\pi}{5})S_{\infty\infty}^{(1)}(0,5n+1,c,\mu_5) + C_5^{4,2}\sin(\tfrac{2\pi}{5})S_{\infty\infty}^{(2)}(0,5n+1,c,\mu_5) = 0, \tag{5-1}$$

$$C_5^{0,4}\sin(\tfrac{\pi}{5})S_{\infty\infty}^{(1)}(0,5n+2,c,\mu_5) + C_5^{0,2}\sin(\tfrac{2\pi}{5})S_{\infty\infty}^{(2)}(0,5n+2,c,\mu_5) = 0, \tag{5-2}$$

$$C_7^{4,6}S_7^{(1)}(7n,c) + C_7^{6,2}S_7^{(2)}(7n,c) + C_7^{2,4}S_7^{(3)}(7n,c) = 0, \tag{7-0}$$

$$C_7^{2,4}S_7^{(1)}(7n+1,c) + C_7^{4,6}S_7^{(2)}(7n+1,c) + C_7^{6,2}S_7^{(3)}(7n+1,c) = 0, \tag{7-1,1}$$

$$C_7^{4,6}S_7^{(1)}(7n+1,c) + C_7^{6,2}S_7^{(2)}(7n+1,c) + C_7^{2,4}S_7^{(3)}(7n+1,c) = 0, \tag{7-1,2}$$

$$C_7^{0,6}S_7^{(1)}(7n+2,c) + C_7^{0,2}S_7^{(2)}(7n+2,c) + C_7^{0,4}S_7^{(3)}(7n+2,c) = 0, \tag{7-2}$$

$$C_7^{0,4}S_7^{(1)}(7n+3,c) + C_7^{0,6}S_7^{(2)}(7n+3,c) + C_7^{0,2}S_7^{(3)}(7n+3,c) = 0, \tag{7-3,1}$$

$$C_7^{2,6}S_7^{(1)}(7n+3,c) + C_7^{4,2}S_7^{(2)}(7n+3,c) + C_7^{6,4}S_7^{(3)}(7n+3,c) = 0, \tag{7-3,2}$$

$$C_7^{0,2}S_7^{(1)}(7n+4,c) + C_7^{0,4}S_7^{(2)}(7n+4,c) + C_7^{0,6}S_7^{(3)}(7n+4,c) = 0, \tag{7-4,1}$$

$$C_7^{2,6}S_7^{(1)}(7n+4,c) + C_7^{4,2}S_7^{(2)}(7n+4,c) + C_7^{6,4}S_7^{(3)}(7n+4,c) = 0, \tag{7-4,2}$$

$$\begin{aligned}\left(C_7^{0,4} + C_7^{2,6}\right)S_7^{(1)}(7n+6,c) + \left(C_7^{0,6} + C_7^{4,2}\right)S_7^{(2)}(7n+6,c)&\\ + \left(C_7^{0,2} + C_7^{6,4}\right)S_7^{(3)}(7n+6,c) = 0,&\end{aligned} \tag{7-6}$$

Using (1.5) and Theorem 1.2, we have the following corollary.

Corollary 1.4. *For any pair (p-k) (or (p-k,t) for both* $t = 1, 2$*) in Theorem 1.3 with*

$$p = 5,\ k \in \{1,2,4\} \quad or \quad p = 7,\ k \in \{0,1,2,3,4,5,6\},$$

we have Dyson's conjecture (p-k) in Theorem 1.1.

The paper is organized as follows. In Section 2 we review about our notations of vector-valued Kloosterman sums as in [Sun24]. In Section 3 we give the detailed proof of (5-4) of Theorem 1.3. In Section 4 and Section 5 we prove (7-5) of Theorem 1.3. Section 6 is the proof of the remaining part, i.e. part (ii) of Theorem 1.3.

2. Notations

In this section we define some notation involving Dedekind sums and Kloosterman sums. For the origin of these notations, see [Gar17, Sun24].

For integers d and $m \geq 1$, let $\overline{d_{\{m\}}}$ denote the inverse of d (mod m). If there is a subscript, e.g. d_1, then we write $\overline{d_{1\{m\}}}$ as the inverse of d_1 (mod m).

Define

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2}, & \text{when } x \in \mathbb{R}\setminus\mathbb{Z},\\ 0, & \text{when } x \in \mathbb{Z}.\end{cases}$$

For integers $c>0$ and $(d,c)=1$, we define the Dedekind sum as

$$s(d,c):=\sum_{r \pmod c}\left(\left(\frac{r}{c}\right)\right)\left(\left(\frac{dr}{c}\right)\right). \tag{2.1}$$

Here we use the notation $r \pmod c$ to indicate that the summation is taken over all residue classes modulo c. Similarly, $r \pmod c)^*$ denotes that the summation is restricted to reduced residue classes, where $\gcd(r,c)=1$. For simplicity in more complex subscripts, we will abbreviate these as $r(c)$ and $r(c)^*$, respectively.

The Dedekind sums have the following properties [Lew95, (4.2)-(4.5)]:

$$2\theta cs(d,c)\in\mathbb{Z},\quad \text{where } \theta=\gcd(c,3), \tag{2.2}$$

$$12cs(d,c)\equiv d+\overline{d_{\{\theta c\}}} \pmod{\theta c}, \tag{2.3}$$

$$12cs(d,c)\equiv c+1-2(\tfrac{d}{c}) \pmod 8,\quad \text{if } c \text{ is odd}, \tag{2.4}$$

$$12cs(d,c)\equiv d+\big(c^2+3c+1+2c(\tfrac{c}{d})\big)\,\overline{d_{\{8\times 2^\lambda\}}} \pmod{8\times 2^\lambda},\quad \text{if } 2^\lambda\| c \text{ for } \lambda\ge 1. \tag{2.5}$$

These congruences determine $12cs(d,c) \pmod{24c}$ uniquely in every case ($2|c$ or $2\nmid c$, $3|c$ or $3\nmid c$).

In the proof we use the following quadratic reciprocity of the Kronecker symbol $(\frac{\cdot}{\cdot})$. For any non-zero integer n, write $n=2^\lambda n_o$ where n_o is odd. For integers m,n with $(m,n)=1$, we have

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right)=\pm(-1)^{(m_o-1)(n_o-1)/4}, \tag{2.6}$$

where we take $+$ if $m\ge 0$ or $n\ge 0$, and we take $-$ if $m<0$ and $n<0$.

Next we define the Kloosterman sums $S^{(\ell)}_{\infty\infty}(0,n,c,\mu_p)$ for $p=5,7$ and $S^{(\ell)}_{0\infty}(0,n,c,\mu_7;0)$ appearing at Corollary 1.2. We follow the notations of vector-valued Kloosterman sums in [Sun24, §4.3]. From [Sun24, (5.19), (5.29)], when $p|c$ we have

$$S^{(\ell)}_{\infty\infty}(0,n,c,\mu_p)=e(-\tfrac18)\sum_{\substack{d \pmod c)^*\\ ad\equiv 1 \pmod c}}\frac{(-1)^{\ell c}e(-\frac{3ca\ell^2}{2p^2})}{\sin(\frac{\pi a\ell}{p})}e^{-\pi i s(d,c)}e\left(\frac{nd}{c}\right). \tag{2.7}$$

When $p=7$, recall that $[A\ell]$ is the least non-negative residue of $A\ell \pmod 7$. From [Sun24, (5.31)], when $A\ell=7T+1$ for some integer $T\ge 0$, we have

$$S^{(\ell)}_{0\infty}(0,n,A,\mu_7;0)=(-1)^{A\ell-[A\ell]}\sum_{\substack{B \pmod A)^*\\ 0<C<7A,\,7|C\\ BC\equiv -1(A)}}e\left(\frac{(\frac32T^2+\frac12T)C}{A}\right)e^{-\pi is(B,A)}e\left(\frac{nB}{A}\right). \tag{2.8}$$

When $A\ell=7T-1$ for some integer $T\ge 1$, we have

$$\begin{aligned}&S^{(\ell)}_{0\infty}(0,n,A,\mu_7;0)\\ &=(-1)^{A\ell-[A\ell]}\sum_{\substack{B \pmod A)^*\\ 0<C<7A,\,7|C\\ BC\equiv -1(A)}}e\left(\frac{(\frac32(T-1)^2+\frac52(T-1)+1)C}{A}\right)e^{-\pi is(B,A)}e\left(\frac{nB}{A}\right).\end{aligned} \tag{2.9}$$

If $[A\ell]\neq 1$ and $[A\ell]\neq 6$, then $S^{(\ell)}_{0\infty}(0,n,A,\mu_7;0):=0$.

3. Proof of (5-4) of Theorem 1.3

In this section we prove (5-4) of Theorem 1.3. We only consider $\ell = 1, 2$ because $A(\frac{\ell}{p}; n) = A(1 - \frac{\ell}{p}; n)$.

Define $c' := c/5$. For any integer r with $(r, c') = 1$, we define

$$V(r,c) := \{d \pmod{c}^* : \ d \equiv r \pmod{c'}\}.$$

For example, $V(1,30) = \{d \pmod{30}^* : d \equiv 1, 7, 13, 19 \pmod{30}\}$ and $V(4,25) = \{d \pmod{25}^* : d \equiv 4, 9, 14, 19, 24 \pmod{25}\}$. Clearly, $|V(r,c)| = 4$ if $5\|c$ and $|V(r,c)| = 5$ if $25|c$. Moreover, $(\mathbb{Z}/c\mathbb{Z})^*$ is the disjoint union

$$(\mathbb{Z}/c\mathbb{Z})^* = \bigcup_{r \pmod{c'}^*} V(r,c).$$

By (2.7) we have

$$e(\tfrac{1}{8}) S^{(\ell)}_{\infty\infty}(0, 5n+4, c, \mu_5) = \sum_{\substack{d \pmod{c}^* \\ ad \equiv 1 \pmod{c}}} \frac{(-1)^{\ell c} e(-\frac{3c' a\ell^2}{10})}{\sin(\frac{\pi a \ell}{5})} e^{-\pi i s(d,c)} e\left(\frac{(5n+4)d}{c}\right). \tag{3.1}$$

We claim the following proposition.

Proposition 3.1. *For $\ell = 1, 2$, the sum on $V(r,c)$ satisfies*

$$s_{r,c} := \sum_{\substack{d \in V(r,c) \\ ad \equiv 1 \pmod{c}}} \frac{e(-\frac{3c' a\ell^2}{10})}{\sin(\frac{\pi a \ell}{5})} e^{-\pi i s(d,c)} e\left(\frac{4d}{c}\right) = 0. \tag{3.2}$$

If Proposition 3.1 is true, then

$$S^{(\ell)}_{\infty\infty}(0, 5n+4, c, \mu_5) = e(-\tfrac{1}{8})(-1)^{\ell c} \sum_{r \pmod{c'}^*} s_{r,c}\, e\left(\frac{nr}{c'}\right) = 0$$

for all $n \in \mathbb{Z}$, $\ell = 1, 2$, and we have proved (5-4) of Theorem 1.3.

In the following subsections §7.1-§7.4, we prove Proposition 3.1 when $5\|c$. In §7.5, we prove Proposition 3.1 when $25|c$. Suppose now that $5\|c$. Since $|V(r,c)| = 4$, let $\beta \in \{1,2,3,4\}$ such that $\beta c' \equiv 1 \pmod{5}$ and we make a special choice of $V(r,c)$ as

$$V(r,c) = \{d_1, d_2, d_3, d_4\} \quad \text{where } d_j \equiv j \pmod{5} \text{ and } d_{j+1} = d_1 + j\beta c'. \tag{3.3}$$

We also take a_j for $j \in \{1,2,3,4\}$ such that $a_j \equiv j \pmod{5}$, $a_{j+1} = a_1 + j\beta c'$, and

$$a_{\overline{j_{\{5\}}}} d_j \equiv 1 \pmod{c}. \tag{3.4}$$

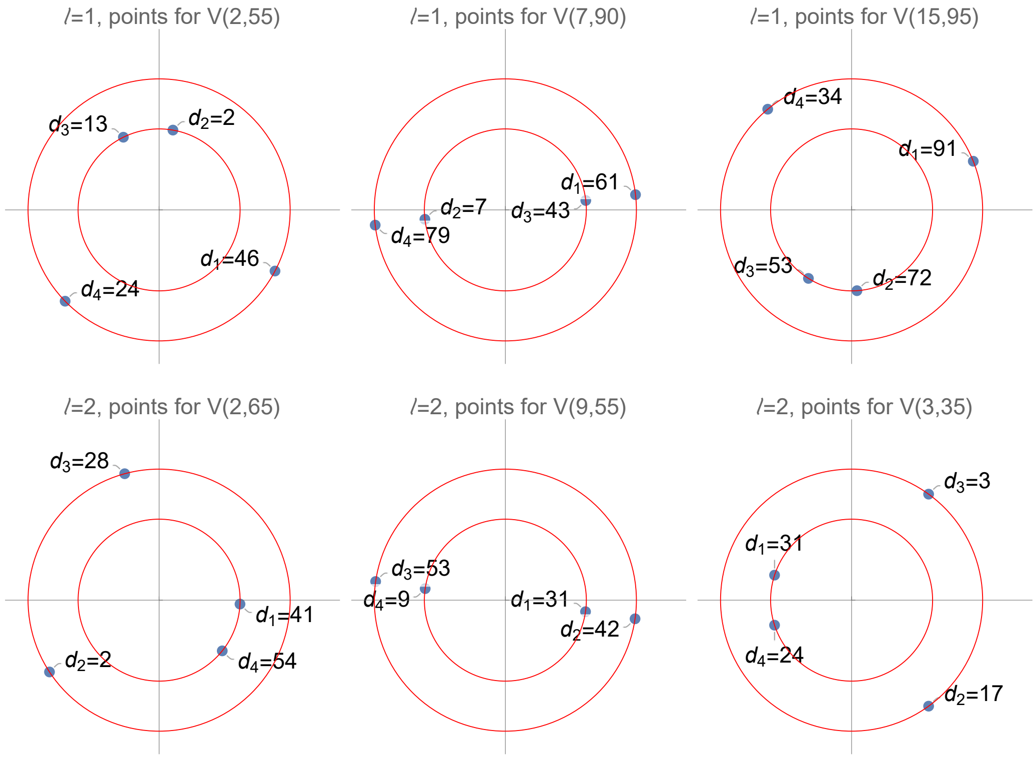
These choices do not affect the sum (3.2) because $s_{r,c}$ has period c in both a and d. In (3.2), we denote each summation term as

$$P(d) := \frac{e\left(-\frac{3c' a\ell^2}{10}\right)}{\sin(\frac{\pi a\ell}{5})} \cdot e\left(-\frac{12cs(d,c)}{24c}\right) \cdot e\left(\frac{4d}{c}\right) =: P_1(d) \cdot P_2(d) \cdot P_3(d), \tag{3.5}$$

where $P_1(d) := e(-\frac{3c' a\ell^2}{10}) / \sin(\frac{\pi a\ell}{5})$, $P_2(d) := \exp(-\pi i s(d,c))$, and $P_3(d) := e(\frac{4d}{c})$.

Remark. We keep $24c$ in the denominator of $P_2(d)$ because the congruence properties of the Dedekind sum are of the form $12cs(d,c)$. See (2.2)-(2.5) for details.

We claim that the set of points $P(d)$ for $d \in V(r,c)$ must have the relative position illustrated in one of the following six configurations. Here $0 < d_j < c$ for simplicity but we use (3.3) in the proof.



Here we explain the styles. Each graph above has two circles with inner one of radius $\csc(\frac{2\pi}{5})$ and outer one with radius $\csc(\frac{\pi}{5})$. When $\ell = 1$, the value of $P(d_1)$ and $P(d_4)$ will be on the outer circle ($P(d_2)$ and $P(d_3)$ on the inner circle) because the term $P_1(d_j)$ has denominator $\sin(\frac{\pi a_j \ell}{5})$. When $\ell = 2$, $P(d_1)$ and $P(d_4)$ will be on the inner circle.

We describe the relative argument differences via the following notation. Let

$$\mathrm{Arg}_j(d_u \to d_v; \ell), \qquad \text{for } j \in \{1,2,3\},\ u,v \in \{1,2,3,4\}, \text{ and } \ell \in \{1,2\} \tag{3.6}$$

be the argument difference (as the proportion of 2π, positive when going counter-clockwise) contributed from P_j going from d_u to d_v when $\ell \in \{1,2\}$. To be precise, if we denote $P_j(d_u) = R_{j,u} \exp(i\Theta_{j,u})$ for $R_{j,u}, \Theta_{j,u} \in \mathbb{R}$, then

$$\mathrm{Arg}_j(d_u \to d_v; \ell) = \alpha \quad \Leftrightarrow \quad \Theta_{j,v} - \Theta_{j,u} = \alpha \cdot 2\pi + 2k\pi \text{ for some } k \in \mathbb{Z}.$$

We say that two argument differences equal: $\mathrm{Arg}_j(d_u \to d_v; \ell) = \mathrm{Arg}_j(d_w \to d_x; \ell)$ or say $\mathrm{Arg}_j(d_u \to d_v; \ell) = \alpha$ if their difference is an integer.

Although the P_2 and P_3 terms are not affected by the value of ℓ in (3.5), we still use the notations $\mathrm{Arg}_2(d_u \to d_v; \ell)$ and $\mathrm{Arg}_e(d_u \to d_v; \ell)$ to indicate the different cases for ℓ. Moreover, we define

$$\mathrm{Arg}(d_u \to d_v; \ell) := \sum_{j=1}^{3} \mathrm{Arg}_j(d_u \to d_v; \ell) \tag{3.7}$$

as the argument difference in total.

The following condition ensures Proposition 3.1.

Condition 3.2. *We have the following six styles for the relative position of these four points.*

- $\ell = 1$. *First graph style: the arguments going $d_1 \to d_2 \to d_3 \to d_4 \to d_1$ are $\frac{3}{10}$, $\frac{1}{10}$, $\frac{3}{10}$, and $\frac{3}{10}$, respectively. The second graph style is that all the argument differences are $\frac{1}{2}$, while the third graph style has the reversed order of rotation compared with the first one.*

$\mathrm{Arg}(d_u \to d_v; 1) \searrow$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$c' \equiv 1 \pmod 5$		$\frac{3}{10}$		$\frac{1}{10}$		$\frac{3}{10}$		$\frac{3}{10}$	
$c' \equiv 2, 3 \pmod 5$		$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$		$\frac{1}{2}$	
$c' \equiv 4 \pmod 5$		$-\frac{3}{10}$		$-\frac{1}{10}$		$-\frac{3}{10}$		$-\frac{3}{10}$	

- $\ell = 2$. *Here are the styles for the graphs in the second row.*

$\mathrm{Arg}(d_u \to d_v; 2) \searrow$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$c' \equiv 3 \pmod 5$		$-\frac{2}{5}$		$-\frac{3}{10}$		$-\frac{2}{5}$		$\frac{1}{10}$	
$c' \equiv 1, 4 \pmod 5$		0		$\frac{1}{2}$		0		$\frac{1}{2}$	
$c' \equiv 2 \pmod 5$		$\frac{2}{5}$		$\frac{3}{10}$		$\frac{2}{5}$		$-\frac{1}{10}$	

One can check that, whenever the four points on $\mathbb{C}$ satisfy any of the above cases of relative argument differences and corresponding radii, their sum becomes 0. This can be explained by

$$\frac{\cos(\frac{\pi}{10})}{\sin(\frac{2\pi}{5})} = \frac{\cos(\frac{3\pi}{10})}{\sin(\frac{\pi}{5})} = 1, \quad \text{where } \frac{1}{\sin(\frac{\pi}{5})} \text{ and } \frac{1}{\sin(\frac{2\pi}{5})} \text{ are the radii.}$$

Before we divide into the cases, we first claim the following lemma:

Lemma 3.3. *For $\ell \in \{1, 2\}$, we have*

$$\mathrm{Arg}(d_1 \to d_2; \ell) + \mathrm{Arg}(d_4 \to d_3; \ell) = 0 \quad \textit{and} \quad \mathrm{Arg}(d_1 \to d_3; \ell) + \mathrm{Arg}(d_4 \to d_2; \ell) = 0. \quad (3.8)$$

Granted the above reduction, to prove that each case of the argument differences are one of the cases in Condition 3.2, we only need to verify that

$$\mathrm{Arg}(d_1 \to d_4; \ell) \quad \text{and} \quad \mathrm{Arg}(d_1 \to d_2; \ell) \quad \text{for } \ell = 1, 2$$

satisfy Condition 3.2. We prove this by enumerating all of the cases. We can list the argument differences for Arg_1 and Arg_3, but for Arg_2, we require the congruence properties of Dedekind sums in (2.2)-(2.5).

Proof of Lemma 3.3. Note that

$$\mathrm{Arg}(d_u \to d_v; \ell) = \mathrm{Arg}(d_u \to d_w; \ell) + \mathrm{Arg}(d_w \to d_v; \ell)$$

for all $u, v, w \in \{1, 2, 3, 4\}$. Then it suffices to prove

$$\mathrm{Arg}(d_1 \to d_2; \ell) = \mathrm{Arg}(d_3 \to d_4; \ell).$$

Recall our notation for d_j and a_j in (3.3). Since $a_3 - a_1 = a_4 - a_2 = 2\beta c'$, one can show $\mathrm{Arg}_1(d_1 \to d_2; \ell) = \mathrm{Arg}_1(d_3 \to d_4; \ell)$ by

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_3 \ell}{5}) / \sin(\tfrac{\pi a_1 \ell}{5})\right) = \mathrm{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5}) / \sin(\tfrac{\pi a_2 \ell}{5})\right) = 1.$$

It is also easy to show $\mathrm{Arg}_3(d_1 \to d_2; \ell) = \mathrm{Arg}_3(d_3 \to d_4; \ell)$.

For Arg_2, we apply (2.3), (2.4) and (2.5) with the Chinese Remainder Theorem to show

$$12cs(d_2, c) - 12cs(d_1, c) \equiv 12cs(d_4, c) - 12cs(d_3, c) \pmod{24c}$$

in the following cases.

When $\gcd(c,3)=1$, we recall $d_2-d_1=d_4-d_3=\beta c'$ and have

$$12cs(d_2,c)-12cs(d_1,c)\equiv d_2+a_3-d_1-a_1\equiv 3\beta c' \pmod{c}, \tag{3.9}$$
$$12cs(d_4,c)-12cs(d_3,c)\equiv d_4+a_4-d_3-a_2\equiv 3\beta c' \pmod{c}, \tag{3.10}$$
$$12cs(d_2,c)-12cs(d_1,c)\equiv 12cs(d_4,c)-12cs(d_3,c)\equiv 0 \pmod{6}. \tag{3.11}$$

When $3|c$, we apply the congruence

$$\overline{(x+y)_{\{m\}}}-\overline{x_{\{m\}}}\equiv -y\overline{(x+y)_{\{m\}}}\cdot\overline{x_{\{m\}}} \pmod{m} \tag{3.12}$$

to compute

$$d_2+\overline{d_{2\{3c\}}}-d_1-\overline{d_{1\{3c\}}}\equiv \beta c'(1-\overline{d_{2\{3c\}}}\cdot\overline{d_{1\{3c\}}}) \pmod{3c},$$
$$d_4+\overline{d_{4\{3c\}}}-d_3-\overline{d_{3\{3c\}}}\equiv \beta c'(1-\overline{d_{4\{3c\}}}\cdot\overline{d_{3\{3c\}}}) \pmod{3c},$$

which imply

$$12cs(d_2,c)-12cs(d_1,c)\equiv 12cs(d_4,c)-12cs(d_3,c)\equiv 0 \pmod{c'}$$

by (2.3). After dividing by c' (recall that the denominator of $P_2(d)$ is $24c$), we have

$$60s(d_2,c)-60s(d_1,c)\equiv \beta(1-\overline{d_{2\{3c\}}}\cdot\overline{d_{1\{3c\}}})\equiv\beta(1-a_3a_1) \pmod{15}, \tag{3.13}$$
$$60s(d_4,c)-60s(d_3,c)\equiv \beta(1-\overline{d_{4\{3c\}}}\cdot\overline{d_{3\{3c\}}})\equiv\beta(1-a_4a_2) \pmod{15}, \tag{3.14}$$

because of (3.4) and $\overline{x_{\{un\}}}\equiv\overline{x_{\{vn\}}} \pmod{n}$. Since $a_3\equiv a_1 \pmod 3$ and $a_4\equiv a_2 \pmod 3$, we have $a_3a_1\equiv a_4a_2\equiv 1 \pmod 3$. Moreover, $a_3a_1\equiv a_4a_2\equiv 3 \pmod 5$. Hence $a_3a_1\equiv a_4a_2\equiv 13 \pmod{15}$ and we get

$$60s(d_2,c)-60s(d_1,c)\equiv 60s(d_4,c)-60s(d_3,c)\equiv 3\beta \pmod{15}. \tag{3.15}$$

When c is odd, by (2.4) and $d_{j_1}\equiv d_{j_2} \pmod{c'}$, we have

$$12cs(d_2,c)-12cs(d_1,c)\equiv 2(\tfrac{d_1}{c})-2(\tfrac{d_2}{c})\equiv 2(\tfrac{1}{5})(\tfrac{d_1}{c'})-2(\tfrac{2}{5})(\tfrac{d_1}{c'})\equiv 4 \pmod 8, \tag{3.16}$$
$$12cs(d_4,c)-12cs(d_3,c)\equiv 2(\tfrac{d_3}{c})-2(\tfrac{d_4}{c})\equiv 2(\tfrac{3}{5})(\tfrac{d_3}{c'})-2(\tfrac{4}{5})(\tfrac{d_3}{c'})\equiv 4 \pmod 8. \tag{3.17}$$

When c is even and $2^\lambda\| c$ for $\lambda\ge 1$, by (2.5) and (3.12) we have

$$\begin{aligned}12cs(d_2,c)-12cs(d_1,c)&\equiv d_2+(c^2+3c+1)\overline{d_{2\{8\times 2^\lambda\}}}+2c(\tfrac{c}{d_2})\overline{d_{2\{8\times 2^\lambda\}}}\\&\quad -d_1-(c^2+3c+1)\overline{d_{1\{8\times 2^\lambda\}}}-2c(\tfrac{c}{d_1})\overline{d_{1\{8\times 2^\lambda\}}}\\&\equiv \beta c'(1-(c^2+3c+1)\overline{d_{2\{8\times 2^\lambda\}}}\cdot\overline{d_{1\{8\times 2^\lambda\}}})\\&\quad +2c(\tfrac{c}{d_2})\overline{d_{2\{8\times 2^\lambda\}}}-2c(\tfrac{c}{d_1})\overline{d_{1\{8\times 2^\lambda\}}} \pmod{8\times 2^\lambda}.\end{aligned}$$

Since $12cs(d_2,c)-12cs(d_1,c)$ is a multiple of c' by the discussion of $\gcd(c,3)=1$ or $3|c$ above, by dividing c' and by $x^2\equiv 1 \pmod 8$ for odd x we have

$$60s(d_2,c)-60s(d_1,c)\equiv \beta(1-(c^2+3c+1)d_2d_1)+2(\tfrac{c}{d_2})d_2-2(\tfrac{c}{d_1})d_1 \pmod 8.$$

Similarly, $12cs(d_4,c)-12cs(d_3,c)$ is a multiple of c' and

$$60s(d_4,c)-60s(d_3,c)\equiv \beta(1-(c^2+3c+1)d_4d_3)+2(\tfrac{c}{d_4})d_4-2(\tfrac{c}{d_3})d_3 \pmod 8.$$

Dividing into cases for $4|c$ or $2\|c$ with $c'\equiv 2$ or $6 \pmod 8$, one can conclude

$$d_2d_1\equiv d_4d_3 \pmod 8.$$

For the remaining part, we only need to determine $(\frac{c}{d_j})d_j \equiv \pm 1 \pmod 4$ for $j \in \{1,2,3,4\}$. Since $d_3 \equiv d_1 \pmod 4$ and $d_2 \equiv d_4 \pmod 4$, it is not hard to show that

$$(\tfrac{c}{d_2})d_2 - (\tfrac{c}{d_1})d_1 \equiv (\tfrac{c}{d_4})d_4 - (\tfrac{c}{d_3})d_3 \pmod 4.$$

Now we have proved that when c is even,

$$60s(d_2,c) - 60s(d_1,c) \equiv 60s(d_4,c) - 60s(d_3,c) \pmod 8. \tag{3.18}$$

Combining (3.9), (3.10), (3.11), (3.15), (3.16), (3.17), (3.18), we have shown

$$\mathrm{Arg}_2(d_1 \to d_2;\ell) = \mathrm{Arg}_2(d_3 \to d_4;\ell) \quad \text{for } \ell \in \{1,2\}$$

by proving

$$\frac{12cs(d_2,c) - 12cs(d_1,c)}{24c} - \frac{12cs(d_4,c) - 12cs(d_3,c)}{24c} \in \mathbb{Z}$$

in all the cases for c ($2|c$ or $2 \nmid c$, $3|c$ or $3 \nmid c$). The lemma follows.

□

Now we begin to prove that $\mathrm{Arg}(d_1 \to d_4;\ell)$ and $\mathrm{Arg}(d_1 \to d_2;\ell)$ both satisfy Condition 3.2 in all the cases of $5\|c$.

3.1. Case $2 \nmid c'$, $3 \nmid c'$, and $5 \nmid c'$.

We first treat the case when $c' \equiv 1,7,11,13,17,19,23,29 \pmod{30}$. Recall our notations in (3.3) and (3.4):

$$d_4 = d_1 + 3\beta c', \quad d_2 = d_1 + \beta c', \quad a_4 = a_1 + 3\beta c', \quad a_3 = a_1 + 2\beta c', \quad \beta c' \equiv 1 \pmod 5.$$

The argument differences $\mathrm{Arg}_j(d_1 \to d_4;\ell)$ for $j = 1,2,3$ are given by the arguments of

$$\frac{e\left(-\frac{9}{10}\beta c'^2\ell^2\right)}{\mathrm{sgn}\left(\sin(\frac{\pi a_4 \ell}{5})/\sin(\frac{\pi a_1 \ell}{5})\right)}, \quad e\left(-\frac{12cs(d_4,c) - 12cs(d_1,c)}{24c}\right), \quad \text{and } e\left(\frac{2\beta}{5}\right),$$

respectively. First we have

$$\mathrm{sgn}\left(\sin(\tfrac{\pi a_4 \ell}{5})/\sin(\tfrac{\pi a_1 \ell}{5})\right) = -1 \quad \text{whenever } \begin{cases} \ell = 1 \quad \text{and} \\ 3\beta c' \equiv 8 \pmod{10} \end{cases} \text{or } \ell = 2. \tag{3.19}$$

This is easy to prove because $3\beta c' \times 2 \equiv 6 \pmod{10}$.

By (2.3), we have $\theta = 1$ and

$$-12cs(d_4,c) + 12cs(d_1,c) \equiv -d_4 - a_4 + d_1 + a_1 \equiv -6\beta c' \equiv -\beta c' \pmod c. \tag{3.20}$$

Moreover, we have $-12cs(d_4,c) + 12cs(d_1,c) \equiv 0 \pmod 6$ and

$$-12cs(d_4,c) + 12cs(d_1,c) \equiv 2\left((\tfrac{d_4}{c}) - (\tfrac{d_1}{c})\right) \equiv 2\left((\tfrac{d_4}{5})(\tfrac{d_4}{c'}) - (\tfrac{d_1}{5})(\tfrac{d_1}{c'})\right) \equiv 0 \pmod 8.$$

Here we have used $(\frac{d_j}{5}) = 1$ for $j = 1,4$ and $d_j \equiv d_1 \pmod{c'}$ for all j. Then,

$$-12cs(d_4,c) + 12cs(d_1,c) \equiv 0 \pmod{24}. \tag{3.21}$$

Combining (3.20) and (3.21), since c' is odd, we can divide both the denominator and numerator in P_2 by $24c'$. We obtain

$$\mathrm{Arg}_2(d_1 \to d_4;\ell) = \frac{\beta}{5}.$$

Now we have Table 3.1. In the row of $\mathrm{Arg}_1(d_1 \to d_4;1)$, we see $+\frac{1}{2}$ because the sign difference $\mathrm{sgn}\left(\sin(\frac{\pi a_4 \ell}{5})/\sin(\frac{\pi a_1 \ell}{5})\right) = -1$ when $3\beta c' \equiv 8 \pmod{10}$. The $\mathrm{Arg}_1(d_1 \to d_4;2)$ contains the term $+\frac{1}{2}$ because $3\beta c' \times 2 \equiv 6 \pmod{10}$. The upper half of the table is for the case $\ell = 1$ and the lower half is for $\ell = 2$.

c' (mod 30)	1	7	11	13	17	19	23	29
β	1	3	1	2	3	4	2	4
$3\beta c'$ (mod 10)	3	3	3	$\boxed{8}$	3	$\boxed{8}$	$\boxed{8}$	$\boxed{8}$
$-9\beta c'^2$ (mod 10)	1	7	1	8	7	4	7	4
$\mathrm{Arg}_1(d_1 \to d_4; 1)$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$	$-\frac{2}{10}+\frac{1}{2}$	$-\frac{3}{10}$	$-\frac{6}{10}+\frac{1}{2}$	$-\frac{3}{10}+\frac{1}{2}$	$-\frac{6}{10}+\frac{1}{2}$
$\mathrm{Arg}_2(d_1 \to d_4; 1)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
$\mathrm{Arg}_3(d_1 \to d_4; 1)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 1)$	$-\frac{3}{10}$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{3}{10}$
$3\beta c'$ (mod 10)	3	3	3	8	3	8	8	8
$-18\beta c'^2 \equiv 2c'$ (mod 5)	2	4	2	1	4	3	1	3
$\mathrm{Arg}_1(d_1 \to d_4; 2) : \frac{1}{2}+\frac{2c'}{5}$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$
$\mathrm{Arg}_2(d_1 \to d_4; 2)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
$\mathrm{Arg}_3(d_1 \to d_4; 2)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 2)$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{2}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{2}$	$-\frac{1}{10}$	$\frac{1}{2}$

TABLE 3.1. Table for $\mathrm{Arg}(d_1 \to d_4; \ell)$; $2 \nmid c$, $3 \nmid c$, $5 \nmid c$.

For $\mathrm{Arg}_j(d_1 \to d_2; \ell)$, recall $a_3 d_2 \equiv 1 \pmod{c}$. The argument differences $\mathrm{Arg}_j(d_1 \to d_2; \ell)$ for $j = 1, 2, 3$ are given by

$$\frac{e\left(-\frac{3}{5}\beta c'^2 \ell^2\right)}{\operatorname{sgn}\left(\sin(\frac{\pi a_3 \ell}{5})/\sin(\frac{\pi a_1 \ell}{5})\right)}, \quad e\left(-\frac{12cs(d_2,c) - 12cs(d_1,c)}{24c}\right), \quad e\left(\frac{4\beta}{5}\right),$$

respectively. Since $2\beta c'\ell \equiv 2\ell \pmod{10}$, we always have

$$\operatorname{sgn}\left(\sin(\tfrac{\pi a_3}{5})/\sin(\tfrac{\pi a_1}{5})\right) = 1 \quad \text{and} \quad \operatorname{sgn}\left(\sin(\tfrac{2\pi a_3}{5})/\sin(\tfrac{2\pi a_1}{5})\right) = -1. \tag{3.22}$$

Moreover, from (2.3) we have

$$12cs(d_2,c) - 12cs(d_1,c) \equiv d_2 + a_3 - d_1 - a_1 \equiv 3\beta c' \pmod{c}, \tag{3.23}$$

$$12cs(d_2,c) - 12cs(d_1,c) \equiv -2\left(-(\tfrac{d_2}{c'}) - (\tfrac{d_1}{c'})\right) \equiv 4 \pmod{8}, \tag{3.24}$$

$$12cs(d_2,c) - 12cs(d_1,c) \equiv 0 \pmod{6}, \quad \text{and} \tag{3.25}$$

$$12cs(d_2,c) - 12cs(d_1,c) \equiv 12 \pmod{24}. \tag{3.26}$$

Combining (3.23) and (3.26), we divide by c' and determine the unique value modulo 120:

$$-(60s(d_2,c) - 60s(d_1,c)) \quad \text{congruent to} \quad -3\beta \pmod{5} \quad \text{and} \quad 12 \pmod{24}.$$

This gives the contribution of the argument difference from P_2. Now we can make Table 3.2.

Combining Table 3.1 and Table 3.2, we see that $\mathrm{Arg}(d_1 \to d_4; \ell)$ and $\mathrm{Arg}(d_1 \to d_2; \ell)$ for $\ell = 1, 2$ satisfy the styles in Condition 3.2. This finishes the proof when $2 \nmid c'$, $3 \nmid c'$ and $5 \nmid c'$.

3.2. **Case $2 \nmid c'$, $3 | c'$, and $5 \nmid c'$.** These are the cases when $c' \equiv 3, 9, 21, 27 \pmod{30}$. For $\mathrm{Arg}_1(d_1 \to d_4; \ell)$ we use (3.19). For $\mathrm{Arg}_2(d_1 \to d_4; \ell)$, we need the congruence (3.12). By (2.3), we have

$$12cs(d_4,c) - 12cs(d_1,c) \equiv d_4 + \overline{d_{4\{3c\}}} - d_1 - \overline{d_{1\{3c\}}} \equiv 3\beta c'(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{3.27}$$

c' (mod 30)	1	7	11	13	17	19	23	29
β	1	3	1	2	3	4	2	4
$-3c'$ (mod 5)	2	4	2	1	4	3	1	3
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\mathrm{Arg}_2(d_1 \to d_2; 1)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 1)$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{1}{2}$	$-\frac{3}{10}$
$-3c' \times 4$ (mod 5)	3	1	3	4	1	2	4	2
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} - \frac{12c'}{5}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_2(d_1 \to d_2; 2)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 2)$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	0	$\frac{2}{5}$	0	$-\frac{2}{5}$	$\frac{2}{5}$	0	$-\frac{2}{5}$	0

TABLE 3.2. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2 \nmid c$, $3 \nmid c$, $5 \nmid c$.

By (2.4) we also have

$$12cs(d_4, c) - 12cs(d_1, c) \equiv 0 \pmod 8. \tag{3.28}$$

Dividing the numerator and denominator of P_2 by $24c'$, we observe that

$$-\tfrac{5}{2}(s(d_4, c) - s(d_1, c)) \equiv -\overline{8_{\{5\}}}\beta(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \equiv \beta \pmod 5 \tag{3.29}$$

because $\overline{d_{j\{3c\}}} \equiv \overline{d_{j\{5\}}} \equiv j \pmod 5$ for $j = 1, 4$. Now we get $\mathrm{Arg}_2(d_1 \to d_4; \ell) = \frac{\beta}{5}$. Since $\mathrm{Arg}_3(d_1 \to d_4; \ell) = \frac{2\beta}{5}$, we have Table 3.3.

c' (mod 30)	3	9	21	27
β	2	4	1	3
$3\beta c'$ (mod 10)	8	8	3	3
$-9\beta c'^2$ (mod 10)	8	4	1	7
$\mathrm{Arg}_1(d_1 \to d_4; 1)$	$-\frac{2}{10} + \frac{1}{2}$	$\frac{4}{10} - \frac{1}{2}$	$\frac{1}{10}$	$-\frac{3}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 1)$: $\frac{3\beta}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 1)$	$\frac{1}{2}$	$\frac{3}{10}$	$-\frac{3}{10}$	$\frac{1}{2}$
$3\beta c'$ (mod 10)	8	8	3	3
$-18\beta c'^2$ (mod 5)	1	3	2	4
$\mathrm{Arg}_1(d_1 \to d_4; 2)$	$-\frac{3}{10}$	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 2)$: $\frac{3\beta}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 2)$	$-\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{10}$

TABLE 3.3. Table for $\mathrm{Arg}(d_1 \to d_4; \ell)$; $2 \nmid c$, $3 | c$, $5 \nmid c$.

Next we investigate $\mathrm{Arg}(d_1 \to d_2; \ell)$. For $\mathrm{Arg}_1(d_1 \to d_2; \ell)$, we use (3.22). For $\mathrm{Arg}_2(d_1 \to d_2; \ell)$, by (2.3) we have

$$12cs(d_2, c) - 12cs(d_1, c) \equiv d_2 + \overline{d_{2\{3c\}}} - d_1 - \overline{d_{1\{3c\}}} \equiv \beta c'(1 - \overline{d_{2\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{3.30}$$

As $15|3c$, after dividing by c' we have

$$60s(d_2,c)-60s(d_1,c)\equiv\beta(1-\overline{d_{2\{15\}}}\cdot\overline{d_{1\{15\}}})\equiv\beta(1-a_3a_1)\pmod{15}. \tag{3.31}$$

Since $a_3\equiv a_1\pmod 3$, we have $a_3a_1\equiv 1\pmod 3$. We also have $a_3a_1\equiv 3\pmod 5$ by (3.4), then $a_3a_1\equiv 13\pmod{15}$ and

$$-(60s(d_2,c)-60s(d_1,c))\equiv -3\beta\pmod{15}. \tag{3.32}$$

By (2.4) we have

$$12cs(d_4,c)-12cs(d_1,c)\equiv 4\pmod 8. \tag{3.33}$$

The congruences (3.32) and (3.33) determine a unique value modulo 120.

$c'\pmod{30}$	3	9	21	27
β	2	4	1	3
$-3c'\pmod 5$	1	3	2	4
$\operatorname{Arg}_1(d_1\to d_2;1)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{4}{5}$
$\operatorname{Arg}_2(d_1\to d_2;1)$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$
$\operatorname{Arg}_3(d_1\to d_2;1)$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$
Total $\operatorname{Arg}(d_1\to d_2;1)$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{2}$
$-12c'\pmod 5$	4	2	3	1
$\operatorname{Arg}_1(d_1\to d_2;2)$	$\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$
$\operatorname{Arg}_2(d_1\to d_2;2)$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$
$\operatorname{Arg}_3(d_1\to d_2;2)$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$
Total $\operatorname{Arg}(d_1\to d_2;2)$	$-\frac{2}{5}$	0	0	$\frac{2}{5}$

TABLE 3.4. Table for $\operatorname{Arg}(d_1\to d_2;\ell)$; $2\nmid c$, $3|c$, $5\nmid c$.

Combining Table 3.3 and Table 3.4 we finish the proof in the case $2\nmid c'$, $3|c'$ and $5\nmid c'$.

3.3. **Case $2|c'$, $3\nmid c'$, and $5\nmid c'$.** These are the cases $c'\equiv 2,4,8,14,16,22,26,28\pmod{30}$. For $\operatorname{Arg}_1(d_1\to d_4;\ell)$ we still use (3.19). By (2.3), $\theta=1$ and we still have

$$-(12cs(d_4,c)-12cs(d_1,c))\equiv -(d_4+a_4-d_1-a_1)\equiv -6\beta c'\equiv -\beta c'\pmod c, \tag{3.34}$$

and $12cs(d,c)\equiv 0\pmod 6$. Define the integer $\lambda\ge 1$ by $2^\lambda\|c$. To determine the value modulo $24c$, we need to determine it modulo $8\times 2^\lambda$. By (2.5) we have

$$\begin{aligned}12cs(d_4,c)-12cs(d_1,c)&\equiv d_4-d_1+(c^2+3c+1)(\overline{d_{4\{8\times 2^\lambda\}}}-\overline{d_{1\{8\times 2^\lambda\}}})\\&\quad+2c\left(\overline{d_{4\{8\times 2^\lambda\}}}(\tfrac{c}{d_4})-\overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})\right)\pmod{8\times 2^\lambda}\\&\equiv 3\beta c'\left(1-(c^2+3c+1)\overline{d_{4\{8\times 2^\lambda\}}}\cdot\overline{d_{1\{8\times 2^\lambda\}}}\right)\\&\quad+2c\left(\overline{d_{4\{8\times 2^\lambda\}}}(\tfrac{c}{d_4})-\overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1})\right)\pmod{8\times 2^\lambda}.\end{aligned} \tag{3.35}$$

We claim that

$$12cs(d_4,c)-12cs(d_1,c)\equiv 0\pmod{8\times 2^\lambda}. \tag{3.36}$$

To see this, since $2^\lambda \| c'$ and $c' | (12cs(d_4,c) - 12cs(d_1,c))$ by (3.34), we divide (3.35) by c' and obtain

$$\begin{aligned} 60\left(s(d_4,c)-s(d_1,c)\right) &\equiv 3\beta\left(1-(c^2+3c+1)d_4d_1\right)+2\left(d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1})\right)\\ &\equiv 3\beta c'\left(3\beta d_1-1\right)(c'-1)+2\left(d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1})\right) \pmod 8. \end{aligned}$$

Define val:= $3\beta c'\left(3\beta d_1-1\right)(c'-1) \pmod 8$ as the first part of the congruence above. Note that both d_1 and $c'-1$ are odd. We have Table 3.5 for val.

$c' \pmod 5$	1	2	3	4
β	1	3	2	4
$3\beta c'$	$3c'$	$6c'$	$9c'$	$12c'$
$3\beta d_1 - 1 \pmod 2$	$3d_1-1$	$6d_1-1$	$9d_1-1$	$12d_1-1$
$2\|c$, $d_1 \equiv 1 \pmod 4$	4	4	0	0
$2\|c$, $d_1 \equiv 3 \pmod 4$	0	4	4	0
$4\|c$;	0	0	0	0

Table 3.5. Table of val:= $3\beta c'\left(3\beta d_1-1\right)(c'-1) \pmod 8$; $2|c$, no requirement for $(c,3)$, $5\nmid c$.

For the second part we only need to determine $d_4(\frac{c}{d_4})-d_1(\frac{c}{d_1}) \pmod 4$. When $4|c$, we get $(\frac{2^\lambda}{d_4})=(\frac{2^\lambda}{d_1})=1$. By quadratic reciprocity,

$$\begin{aligned} d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1}) &\equiv d_1\left((\tfrac{5}{d_4})\left(\tfrac{c'/2^\lambda}{d_4}\right)-(\tfrac{5}{d_4})\left(\tfrac{c'/2^\lambda}{d_1}\right)\right)\\ &\equiv d_1\left(\tfrac{d_1}{c'/2^\lambda}\right)\left((-1)^{(d_4-1)(\frac{c'}{2^\lambda}-1)/4}-(-1)^{(d_1-1)(\frac{c'}{2^\lambda}-1)/4}\right)\equiv 0 \pmod 4) \end{aligned}$$

where the last equality follows since $\frac{d_4-1}{2}$ and $\frac{d_1-1}{2}$ have the same parity. This gives the last row in Table 3.5.

When $2\|c$, recall that $d_4 = d_1 + 3\beta c'$, from which

$$d_4(\tfrac{c}{d_4})-d_1(\tfrac{c}{d_1}) \equiv (\tfrac{d_1}{c'/2})\left((\tfrac{2}{d_4})(-1)^{(d_4-1)(\frac{c'}{2}-1)/4}d_4-(\tfrac{2}{d_1})(-1)^{(d_1-1)(\frac{c'}{2}-1)/4}d_1\right) \pmod 4 \quad (3.37)$$

When $c' \equiv 2 \pmod 8$, $\frac{c'/2-1}{2}$ is even and (3.37) becomes $(\frac{2}{d_4})d_4-(\frac{2}{d_1})d_1 \pmod 4$; when $c' \equiv 6 \pmod 8$, $\frac{c'/2-1}{2}$ is odd and (3.37) becomes $(\frac{2}{d_4})(-1)^{\frac{d_4-1}{2}}d_4-(\frac{2}{d_1})(-1)^{\frac{d_1-1}{2}}d_1 \pmod 4$. Since $c = 5c' \equiv c' \pmod 8$, we can use $d_4 = d_1 + 3\beta c'$ to determine $d_4 \pmod 8$ and get Table 3.6.

(3.37) ↘	$c' \equiv 2 \pmod 8$				$c' \equiv 6 \pmod 8$			
$d_1 \pmod 8$	1	3	5	7	1	3	5	7
$\beta = 1$	2	0	2	0	2	0	2	0
$\beta = 2$	2	2	2	2	2	2	2	2
$\beta = 3$	0	2	0	2	0	2	0	2
$\beta = 4$	0	0	0	0	0	0	0	0

Table 3.6. Table for (3.37); $2|c$, no requirement for $(c,3)$, $5\nmid c$.

Combining Table 3.5 and Table 3.6, we prove (3.36). Recall (2.2) and (3.34), we divide both the denominator and numerator in P_2 by $24c'$ and get $\mathrm{Arg}_2(d_1 \to d_4; \ell) = \frac{\beta}{5}$. Since $\mathrm{Arg}_3(d_1 \to d_4; \ell) = \frac{2\beta}{5}$, we have Table 3.7.

c' (mod 30)	2	4	8	14	16	22	26	28
β	3	4	2	4	1	3	1	2
$3\beta c'$ (mod 10)	8	8	8	8	8	8	8	8
$-9\beta c'^2$ (mod 10)	2	4	8	4	6	2	6	8
$\mathrm{Arg}_1(d_1 \to d_4; 1)$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 1)$: $\frac{3\beta}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 1)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{3}{10}$	$-\frac{3}{10}$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{1}{2}$
$-18\beta c'^2 \equiv 2c'$ (mod 5)	4	3	1	3	2	4	2	1
$\mathrm{Arg}_1(d_1 \to d_4; 2)$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 2)$: $\frac{3\beta}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 2)$	$\frac{1}{10}$	$\frac{1}{2}$	$-\frac{1}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{2}$	$-\frac{1}{10}$

Table 3.7. Table for $\mathrm{Arg}(d_1 \to d_4; 2)$; $2|c$, $3 \nmid c$, $5 \nmid c$.

Next we deal with $\mathrm{Arg}(d_1 \to d_2; \ell)$. For $\mathrm{Arg}_1(d_1 \to d_2; \ell)$, we still use (3.22). By (2.3),

$$-\left(12cs(d_2, c) - 12cs(d_1, c)\right) \equiv -(d_2 + a_3 - d_1 - a_1) \equiv -3\beta c' \equiv 2\beta c' \pmod{c}. \tag{3.38}$$

This congruence shows that $12cs(d_2, c) - 12cs(d_1, c)$ is divisible by c'. Denote λ by $2^\lambda \| c$. We claim that

$$-\left(12cs(d_2, c) - 12cs(d_1, c)\right) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}. \tag{3.39}$$

To prove (3.39), we apply (2.5) to get

$$\begin{aligned} 12cs(d_2, c) - 12cs(d_1, c) \equiv \beta c' \left(1 - (c^2 + 3c + 1)\overline{d_{4\{8\times 2^\lambda\}}} \cdot \overline{d_{1\{8\times 2^\lambda\}}}\right) \\ + 2c\left(\overline{d_{2\{8\times 2^\lambda\}}}\left(\tfrac{c}{d_2}\right) - \overline{d_{1\{8\times 2^\lambda\}}}\left(\tfrac{c}{d_1}\right)\right) \pmod{8 \times 2^\lambda}. \end{aligned} \tag{3.40}$$

Then as in (3.35), we have

$$60s(d_2, c) - 60s(d_1, c) \equiv \beta c'(\beta d_1 - 1)(c' - 1) + 2\left(d_2\left(\tfrac{c}{d_2}\right) - d_1\left(\tfrac{c}{d_1}\right)\right) \pmod{8}. \tag{3.41}$$

See Table 3.8 for the first term val:= $\beta c' (\beta d_1 - 1)(c' - 1)$ (mod 8) and note that d_1 and $c' - 1$ are both odd.

c' (mod 5)	1	2	3	4
β	1	3	2	4
$\beta c'$	c'	$3c'$	$2c'$	$4c'$
$\beta d_1 - 1$	$d_1 - 1$	$3d_1 - 1$	$2d_1 - 1$	$4d_1 - 1$
$2 \| c$, $d_1 \equiv 1$ (mod 4)	0	4	4	0
$2 \| c$, $d_1 \equiv 3$ (mod 4)	4	0	4	0
$4 \| c$	0	0	0	0

Table 3.8. Table for val.:= $\beta c' (\beta d_1 - 1)(c' - 1)$ (mod 8); $2|c$, no requirement for $(c, 3)$, $5 \nmid c$.

For the second term $2\left(d_2(\frac{c}{d_2}) - d_1(\frac{c}{d_1})\right) \pmod 8$, we argue as above using the quadratic reciprocity (2.6) and omit the details. Combining (2.2), (3.38) and (3.39), we have

$$-(12cs(d_2,c) - 12cs(d_1,c)) \equiv 12 \times 2^\lambda \pmod{24 \times 2^\lambda}.$$

After dividing c', $-60s(d_2,c) + 60s(d_1,c) \pmod{120}$ is uniquely determined by $2\beta \pmod 5$ and $12 \pmod{24}$. Hence

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{1,7,3,9}{10}, \quad \text{for } \beta = 1,2,3,4, \text{ respectively,}$$

and we get Table 3.9.

$c' \pmod{30}$	2	4	8	14	16	22	26	28
β	3	4	2	4	1	3	1	2
$2\beta c' \pmod{10}$	2	2	2	2	2	2	2	2
$-3\beta c'^2 \equiv 2c' \pmod 5$	4	3	1	3	2	4	2	1
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
$\mathrm{Arg}_2(d_1 \to d_2; 1)$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 1)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{1}{2}$	$-\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{2}$
$-12\beta c'^2 \equiv 3c' \pmod 5$	1	2	4	2	3	1	3	4
$\mathrm{Arg}_1(d_1 \to d_2; 2)$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$	$\frac{3}{10}$
$\mathrm{Arg}_2(d_1 \to d_2; 2)$	$\frac{3}{10}$	$-\frac{1}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{3}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 2)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{3}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$\frac{2}{5}$	0	$-\frac{2}{5}$	0	0	$\frac{2}{5}$	0	$-\frac{2}{5}$

TABLE 3.9. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3 \nmid c$, $5 \nmid c$.

Combining Table 3.7 and Table 3.9, we confirm that Condition 3.2 is satisfied in these cases.

3.4. **Case $2|c'$, $3|c'$, and $5 \nmid c'$.** These are the cases $c' \equiv 6, 12, 18, 24 \pmod{30}$. For $\mathrm{Arg}_1(d_1 \to d_4; \ell)$ we use (3.19). For $\mathrm{Arg}_2(d_1 \to d_4; \ell)$, by (2.3) we have

$$-(12cs(d_4,c) - 12cs(d_1,c)) \equiv -3\beta c'(1 - \overline{d_{4\{3c\}}} \cdot \overline{d_{1\{3c\}}}) \pmod{3c}. \tag{3.42}$$

The proof of (3.36) in the former subsection still works for $3|c$. Then $-(12cs(d_4,c) - 12cs(d_1,c))$ is a multiple of $24c'$. After dividing both the denominator and numerator in P_2 and recalling $\overline{d_{j\{3c\}}} \equiv a_j \equiv j \pmod 5$ for $j = 1, 4$, we get $\mathrm{Arg}_2(d_1 \to d_4; \ell) = e(\frac{\beta}{5})$. This gives Table 3.10.

Then we check $\mathrm{Arg}(d_1 \to d_2; \ell)$. For $\mathrm{Arg}_1(d_1 \to d_2; \ell)$, we use (3.22). For $\mathrm{Arg}_2(d_1 \to d_2; \ell)$, by (2.3) we have

$$-(12cs(d_2,c) - 12cs(d_1,c)) \equiv -\beta c'(1 - \overline{d_{2\{3c\}} d_{1\{3c\}}}) \pmod{3c}. \tag{3.43}$$

Since $3|c$, $\overline{d_{2\{3c\}}} \equiv a_3 \pmod{15}$ and $\overline{d_{1\{3c\}}} \equiv a_1 \pmod{15}$. After dividing by c' we have

$$-(60s(d_2,c) - 60s(d_1,c)) \equiv -\beta(1 - a_3 a_1) \pmod{15}.$$

c' (mod 30)	6	12	18	24
β	1	3	2	4
$3\beta c'$ (mod 10)	8	8	8	8
$-9\beta c'^2$ (mod 10)	6	2	8	4
$\mathrm{Arg}_1(d_1 \to d_4; 1)$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 1) : \frac{3\beta}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 1)$	$-\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{10}$
$-18\beta c'^2 \equiv 2c'$ (mod 5)	2	4	1	3
$\mathrm{Arg}_1(d_1 \to d_4; 2)$	$-\frac{1}{10}$	$\frac{3}{10}$	$-\frac{3}{10}$	$\frac{1}{10}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_4; 2) : \frac{3\beta}{5}$	$\frac{3}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
Total $\mathrm{Arg}(d_1 \to d_4; 2)$	$\frac{1}{2}$	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{1}{2}$

TABLE 3.10. Table for $\mathrm{Arg}(d_1 \to d_4; \ell)$; $2|c$, $3|c$, $5 \nmid c$.

We have $a_3 = a_1 + 2\beta c'$ and $a_3 a_1 \equiv 13 \pmod{15}$, so

$$-\left(60s(d_2, c) - 60s(d_1, c)\right) \equiv -3\beta \pmod{15}. \tag{3.44}$$

Denote λ by $2^\lambda \| c$, then (3.39) still holds since

$$-\left(60s(d_2, c) - 60s(d_1, c)\right) \equiv 4 \pmod{8}. \tag{3.45}$$

By (3.44) and (3.45), we obtain

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{1, 7, 3, 9}{10} \quad \text{for } \beta = 1, 2, 3, 4, \text{ respectively.}$$

This gives Table 3.11.

c' (mod 30)	6	12	18	24
β	1	3	2	4
$-3\beta c'^2 \equiv 2c'$ (mod 5)	2	4	1	3
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{2}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{3}{5}$
$\mathrm{Arg}_2(d_1 \to d_2; 1)$	$\frac{1}{10}$	$\frac{3}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 1)$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{10}$
$-12\beta c'^2 \equiv 3c'$ (mod 5)	3	1	4	2
$\mathrm{Arg}_1(d_1 \to d_2; 2)$	$\frac{1}{10}$	$-\frac{3}{10}$	$\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_2(d_1 \to d_2; 2)$	$\frac{1}{10}$	$\frac{3}{10}$	$-\frac{3}{10}$	$-\frac{1}{10}$
$\mathrm{Arg}_3(d_1 \to d_2; 2)$	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{5}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	0	$\frac{2}{5}$	$-\frac{2}{5}$	0

TABLE 3.11. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3|c$, $5 \nmid c$.

Comparing Table 3.10 and Table 3.11, we have proved that Condition 3.2 is satisfied in these cases.

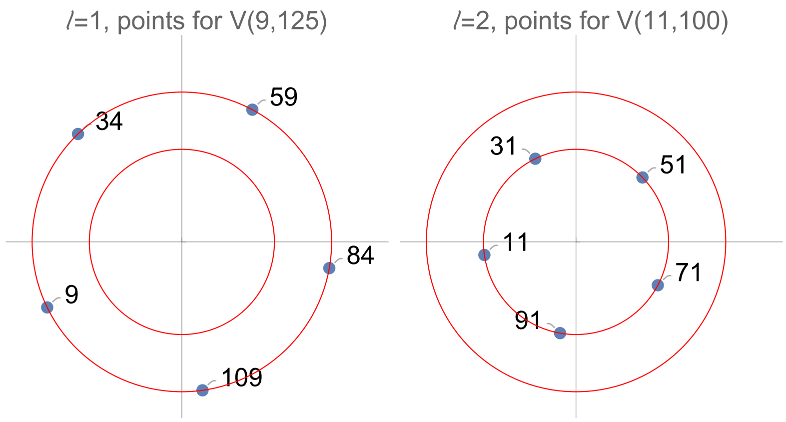
We have finished the proof of Proposition 3.1 by proving that the four points $P(d)$ satisfy Condition 3.2 when $5\|c$. The next subsection is to prove Proposition 3.1 in the case $25|c$, which is different from the former ones.

3.5. **Case** $5|c'$**.** We still denote $c' = c/5$ and $V(r,c) := \{d \pmod{c}^* : \ d \equiv r \pmod{c'}\}$ for $r \pmod{c'}^*$. Now $|V(r,c)| = 5$ and since $(d+c',c)=1$ when $(d,c)=1$, we can write $V(r,c) = \{d, d+c', d+2c', d+3c', d+4c'\}$ for $1 \le d < c'$ and $d \equiv r \pmod{c'}$.

We claim that Proposition 3.1 is still true:

$$\sum_{d\in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{10}\right)}{\sin(\frac{\pi a\ell}{p})} e\left(-\frac{12cs(d,c)}{24c}\right) e\left(\frac{4d}{c}\right) = 0, \tag{3.46}$$

but this time we have five summands. We prove (3.46) by showing that there are only two possible configurations for the summands:



i.e. all at the outer circle (radius $\csc(\frac{\pi}{5})$) or all at the inner circle (radius $\csc(\frac{2\pi}{5})$) and equally distributed. As in (3.5), we still denote the factors in (3.46) by P_1, P_2 and P_3 and investigate the argument differences contributed from each term.

For any $d \in V(r,c)$, we take $a \pmod{c}$ such that $ad \equiv 1 \pmod{c}$. We denote $d_* = d + c'$ and denote a_* by $a_*d_* \equiv 1 \pmod{c}$. Then we can pick $a_* = a - c'$ when $d \equiv 1, 4 \pmod{5}$ and pick $a_* = a + c'$ when $d \equiv 2, 3 \pmod{5}$.

Note that $P_1(d) = (-1)^{ca\ell}/\sin(\frac{\pi a\ell}{5})$ has period c', hence $\mathrm{Arg}_1(d \to d_*; \ell) = 0$ always. In the following two cases, we prove

$$\mathrm{Arg}(d \to d_*; \ell) = -\tfrac{1}{5} \quad \text{for every } d \in V(r,c) \tag{3.47}$$

when $\ell = 1$. The other case $\ell = 2$ only affects P_1 (radii for those five points) and results in the same conclusion. This proves (3.46) when $25|c$.

3.5.1. *c is odd.* When $d \equiv 1, 4 \pmod{5}$ and $3 \nmid c$, (2.2), (2.3) and (2.4) imply that

$$12cs(d_*,c) - 12cs(d,c) \equiv 0 \pmod{24c}, \tag{3.48}$$

hence $\mathrm{Arg}_2(d \to d_*; \ell) = 0$ always. As $\mathrm{Arg}_3(d \to d_*; \ell) = \frac{4}{5}$ for any $d \in V(r,c)$, we have proved (3.46) in this case.

When $3|c$ and $d \equiv 1, 4 \pmod{5}$, (2.3) implies

$$-\left(12cs(d_*,c) - 12cs(d,c)\right) \equiv -c'(1 - \overline{d_{*\{3c\}}} \cdot \overline{d_{\{3c\}}}) \pmod{3c}. \tag{3.49}$$

Since $15|c$, after dividing by c' we have

$$-(60s(d_*,c)-60s(d,c))\equiv a^2-1 \pmod{15}. \tag{3.50}$$

Note that $a\equiv 1,4 \pmod 5$ and $a^2\equiv 1 \pmod{15}$, hence we have $-(12cs(d_*,c)-12cs(d,c))\equiv 0 \pmod{24c}$ and conclude (3.47).

When $d\equiv 2,3 \pmod 5$ and $3\nmid c$, recall $d_*=d+c'$ and $a_1=a+c'$ with $a+d\equiv 0 \pmod 5$. By (2.2), (2.3) and (2.4), we have

$$-(12cs(d_*,c)-12cs(d,c))\equiv -2c' \pmod c \quad \text{and} \quad \equiv 0 \pmod{24}.$$

Then $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{2}{5}$. Since $\mathrm{Arg}_3(d\to d_*;\ell)=\frac{4}{5}$, we have proved (3.47) in this case.

When $d\equiv 2,3 \pmod 5$ and $3|c$, we still get (3.50), while this time $a\equiv 3,2 \pmod 5$, $a^2-1\equiv 3 \pmod{15}$, and hence $a^2-1\equiv 48 \pmod{120}$. We have $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{2}{5}$. Since $\mathrm{Arg}_3(d\to d_*;\ell)=\frac{4}{5}$, we have proved (3.47) in this case.

3.5.2. *c is even.* In this case, denote λ by $2^\lambda\|c$. Then by (2.5) we have

$$\begin{aligned}12cs(d_*,c)-12cs(d,c)\equiv{}& c'\left(1-(c^2+3c+1)\overline{d_{1\{8\times 2^\lambda\}}}\cdot\overline{d_{\{8\times 2^\lambda\}}}\right)\\ &+2c\left(\left(\tfrac{c}{d_*}\right)\overline{d_{1\{8\times 2^\lambda\}}}-\left(\tfrac{c}{d}\right)\overline{d_{\{8\times 2^\lambda\}}}\right) \pmod{8\times 2^\lambda}.\end{aligned}$$

Since $c'|(12cs(d_*,c)-12cs(d,c))$ by (3.48) and (3.49), dividing the above congruence by c' we have

$$-60(s(d_*,c)-s(d,c))\equiv -c'(d-1)(c'-1)-2\left(\left(\tfrac{c}{d_*}\right)d_*-\left(\tfrac{c}{d}\right)d\right) \pmod 8. \tag{3.51}$$

For the first term,

$$-c'(d-1)(c'-1)\equiv\begin{cases}0 \pmod 8 & \text{if } 2\|c,\ d\equiv 1 \pmod 4;\\ 4 \pmod 8 & \text{if } 2\|c,\ d\equiv 3 \pmod 4;\\ 0 \pmod 8 & \text{if } 4|c.\end{cases} \tag{3.52}$$

When λ is even, $\left(\frac{2^\lambda}{d_*}\right)=\left(\frac{2^\lambda}{d}\right)=1$; when $\lambda\ge 3$ is odd, $\left(\frac{2}{d_*}\right)=\left(\frac{2}{d}\right)$. In either case $\frac{d_*-1}{2}$ and $\frac{d-1}{2}$ have the same parity. Hence when $4|c$, we have

$$\left(\tfrac{c}{d_*}\right)d_*-\left(\tfrac{c}{d}\right)d\equiv 0 \pmod 4.$$

When $2\|c$, we have Table 3.12 for val:$=\left(\frac{c}{d_*}\right)d_*-\left(\frac{c}{d}\right)d \pmod 4$ using quadratic reciprocity.

$d \pmod 8$	1	3	5	7
$d_* \pmod 8$ when $c'\equiv 2 \pmod 8$	3	5	7	1
val.	0	2	0	2
$d_* \pmod 8$ when $c'\equiv 6 \pmod 8$	7	1	3	5
val.	0	2	0	2

TABLE 3.12. Table for val:$=\left(\frac{c}{d_*}\right)d_*-\left(\frac{c}{d}\right)d \pmod 4$; $2|c$, no requirement for $(3,c)$, $5|c$.

Combining (3.52) and Table 3.12, for $2^\lambda\|c$ we get

$$12cs(d_*,c)-12cs(d,c)\equiv 0 \pmod{8\times 2^\lambda}. \tag{3.53}$$

The argument for the cases $d\equiv 1,4 \pmod 5$ or $d\equiv 2,3 \pmod 5$, or the cases $3\nmid c$ or $3|c$, still works as the former case.

Proof of Proposition 3.1. This is proved by Condition 3.2 and (3.47). ☐

This finishes the proof of (5-4) in Theorem 1.3.

4. Proof of (7-5,1) of Theorem 1.3

Recall (2.7) in the case $p = 7$:

$$e(\tfrac{1}{8})S^{(\ell)}_{\infty\infty}(0, 7n+5, c, \mu_7) = \sum_{\substack{d \pmod{c}^* \\ ad\equiv 1 \pmod{c}}} \frac{(-1)^{\ell c} e(-\frac{3c'a\ell^2}{14})}{\sin(\frac{\pi a\ell}{7})} e^{-\pi i s(d,c)} e\left(\frac{(7n+5)d}{c}\right). \tag{4.1}$$

We only need to consider $\ell = 1, 2, 3$ because $A(\frac{\ell}{p}; n) = A(1 - \frac{\ell}{p}; n)$.

As in the previous section, we define $c' := c/7$. For an integer r with $(r, c') = 1$, we define

$$V(r, c) = \{d \pmod{c}^* : \ d \equiv r \pmod{c'}\}.$$

For example, $V(1, 42) = \{1, 13, 19, 25, 31, 37\}$ and $V(4, 35) = \{4, 9, 19, 24, 29, 34\}$. Then $|V(r, c)| = 6$ if $7 \nmid c'$, $|V(r, c)| = 7$ if $49|c$, and $(\mathbb{Z}/c\mathbb{Z})^*$ is the disjoint union

$$(\mathbb{Z}/c\mathbb{Z})^* = \bigcup_{r \pmod{c'}^*} V(r, c).$$

We claim the following proposition.

Proposition 4.1. *For $\ell = 1, 2, 3$, when $7|c$, $\frac{c}{7} \cdot \ell \not\equiv 1 \pmod 7$ and $\frac{c}{7} \cdot \ell \not\equiv -1 \pmod 7$, the sum on $d \in V(r, c)$ for all $r \pmod{c'}^*$ is zero:*

$$s_{r,c} := \sum_{d\in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})} e\left(-\frac{12cs(d, c)}{24c}\right) e\left(\frac{5d}{c}\right) = 0 \tag{4.2}$$

If Proposition 4.1 is true, then

$$S^{(\ell)}_{\infty\infty}(0, 7n + 5, c, \mu_7) = e(-\tfrac{1}{8})(-1)^{\ell c} \sum_{r \pmod{c)'}} s_{r,c}\, e\left(\frac{nr}{c'}\right) = 0$$

for all $n \in \mathbb{Z}$, $\ell = 1, 2, 3$ and we have proved (7-5,1) of Theorem 1.3.

As in (3.5), we label the terms in (4.2) as

$$P(d) := \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})} \cdot e\left(-\frac{12cs(d, c)}{24c}\right) \cdot e\left(\frac{5d}{c}\right) =: P_1(d) \cdot P_2(d) \cdot P_3(d). \tag{4.3}$$

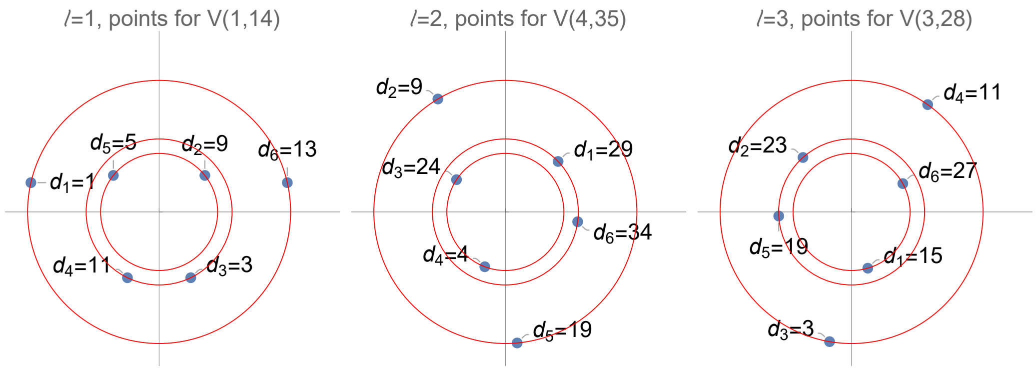
We first deal with the case $7 \nmid c'$. We denote the argument differences as in (3.6), but in this case $u, v \in \{1, 2, \cdots, 6\}$ and $\ell \in \{1, 2, 3\}$, where

$$d_u \equiv a_u \equiv u \pmod 7,\ a_{\overline{u\{7\}}} d_u \equiv 1 \pmod c,\ d_{u+1} = d_u + \beta c' \text{ and } a_{u+1} = a_u + \beta c'. \tag{4.4}$$

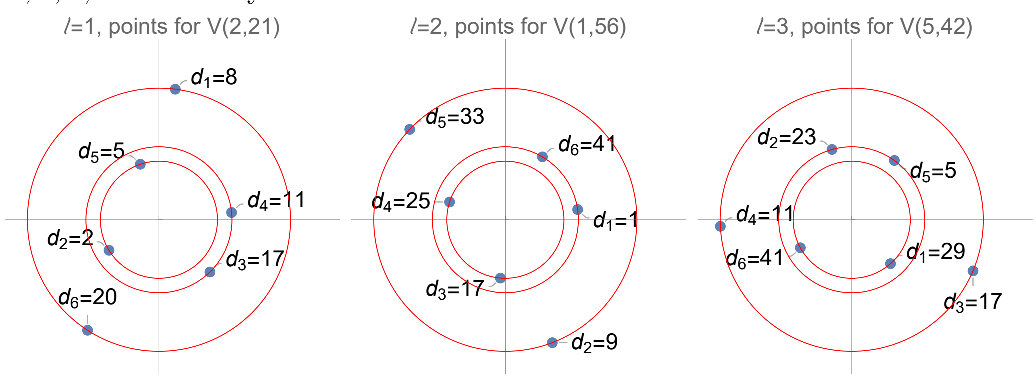
Note that $a_u d_u$ may not be $1 \pmod c$. Let $1 \le \beta \le 6$ such that $\beta c' \equiv 1 \pmod 7$.

As in Condition 3.2, we have the following styles for the six summands followed by the explanation in Condition 4.2:

- $\ell = 1, 2, 3$, first style.



- $\ell = 1, 2, 3$, reversed style from the above.



Here we explain these styles. Each graph above includes three circles centered at the origin with radii $\csc(\frac{\pi}{7})$, $\csc(\frac{2\pi}{7})$ and $\csc(\frac{3\pi}{7})$, respectively. The six points in each graph above mark $P(d)$ for $d \in V(r, c)$ on these three circles. It is not hard to prove that whenever the six points satisfy the following condition on their argument differences, they sum to zero. This proves Proposition 4.1 by using the equation

$$\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})} - \frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})} + \frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})} = 0, \quad \text{where } \frac{1}{\sin(\frac{\pi}{7})}, \frac{1}{\sin(\frac{2\pi}{7})}, \frac{1}{\sin(\frac{3\pi}{7})} \text{ are the radii.}$$

Condition 4.2. *We have the following six styles for these six points when* $7|c$, $\frac{c}{7} \cdot \ell \not\equiv 1 \pmod 7$ *and* $\frac{c}{7} \cdot \ell \not\equiv -1 \pmod 7$.

- $\ell = 1$*: the arguments (as a proportion of* 2π*) going* $d_1 \to d_2 \to d_3 \to d_4 \to d_5 \to d_6 \to d_1$ *are* $-\frac{5}{14}$, $-\frac{2}{7}$, $-\frac{1}{7}$, $-\frac{2}{7}$, $-\frac{5}{14}$, *and* $\frac{3}{7}$, *respectively, or the reversed style.*

	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$c' \equiv 2, 4 \pmod 7$		$-\frac{5}{14}$		$-\frac{2}{7}$		$-\frac{1}{7}$		$-\frac{2}{7}$		$-\frac{5}{14}$		$\frac{3}{7}$	
$c' \equiv 3, 5 \pmod 7$		$\frac{5}{14}$		$\frac{2}{7}$		$\frac{1}{7}$		$\frac{2}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$	

- $\ell = 2$*, second graph style:*

	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$c' \equiv 5, 6 \pmod 7$		$\frac{3}{14}$		$\frac{1}{14}$		$\frac{2}{7}$		$\frac{1}{14}$		$\frac{3}{14}$		$\frac{1}{7}$	
$c' \equiv 1, 2 \pmod 7$		$-\frac{3}{14}$		$-\frac{1}{14}$		$-\frac{2}{7}$		$-\frac{1}{14}$		$-\frac{3}{14}$		$-\frac{1}{7}$	

- $\ell = 3$*, third graph style:*

	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$c' \equiv 1, 4 \pmod 7$		$-\frac{3}{7}$		$\frac{5}{14}$		$\frac{3}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{2}{7}$	
$c' \equiv 3, 6 \pmod 7$		$\frac{3}{7}$		$-\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{5}{14}$		$\frac{3}{7}$		$\frac{2}{7}$	

Proof of Proposition 4.1 when $7 \nmid c'$. This is proved by Condition 4.2. □

Remark. Note that (7-5,1) of Theorem 1.3 is for the case $c'\ell \not\equiv \pm 1 \pmod 7$, so Condition 4.2 does not include all the cases of $c' \pmod 7$. We will highlight these exceptional cases among the tables in this section by a row "$c'\ell \equiv \pm 1 \pmod 7$?". The corresponding entry is:

$$\begin{cases} \text{blank}, & \text{if } c'\ell \not\equiv \pm 1 \pmod 7; \\ \text{“}+\text{”}, & \text{if } c'\ell \equiv 1 \pmod 7; \\ \text{“}-\text{”}, & \text{if } c'\ell \equiv -1 \pmod 7. \end{cases}$$

We will explain these exceptional styles $c'\ell \equiv \pm 1 \pmod 7$ in the next section for (7-5,2).

In the following subsections, we show $\mathrm{Arg}(d_1 \to d_2; \ell)$, $\mathrm{Arg}(d_2 \to d_3; \ell)$, and $\mathrm{Arg}(d_3 \to d_4; \ell)$ in all the cases $c' \pmod{42}$. These argument differences are sufficient to check Condition 4.2 because

$$\mathrm{Arg}(d_1 \to d_2; \ell) = \mathrm{Arg}(d_5 \to d_6; \ell) \text{ and } \mathrm{Arg}(d_2 \to d_3; \ell) = \mathrm{Arg}(d_4 \to d_5; \ell),$$

where the proof is the same as the proof of Lemma 3.3. When $7 \nmid c'$, we prove that $\mathrm{Arg}(d_1 \to d_2; \ell)$, $\mathrm{Arg}(d_2 \to d_3; \ell)$, and $\mathrm{Arg}(d_3 \to d_4; \ell)$ satisfy Condition 4.2 in §4.1-§4.4. When $49|c$, we prove Proposition 4.1 in §4.5.

4.1. **Case $2 \nmid c', 3 \nmid c', 7 \nmid c'$.** We begin by dealing with $\mathrm{Arg}(d_1 \to d_2; \ell)$. First we have

$$\mathrm{Arg}_1(d_1 \to d_2; \ell) = -\frac{9\beta c'^2\ell^2}{14} \begin{cases} +0 & \mathrm{sgn}(\sin(\frac{\pi a_\ell}{7})/\sin(\frac{\pi a_1 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \mathrm{sgn}(\sin(\frac{\pi a_4 \ell}{7})/\sin(\frac{\pi a_1 \ell}{7})) = -1. \end{cases} \tag{4.5}$$

When $\ell = 1$, the sign changes when $3\beta c' \equiv 10 \pmod{14}$. When $\ell = 2$, the sign always changes. When $\ell = 3$, the sign changes when $9\beta c' \equiv 9 \pmod{14}$ but does not change when $9\beta c' \equiv 2 \pmod{14}$.

Since $12cs(d,c) \equiv 0 \pmod 6$, we have

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv -d_2 - a_4 + d_1 + a_1 \equiv -4\beta c' \pmod c,$$

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) - 2(\tfrac{d_1}{7})(\tfrac{d_1}{c'}) \equiv 0 \pmod 8,$$

from which

$$\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{\overline{24_{\{7\}}} \cdot 4\beta}{7} = \frac{\beta}{7}.$$

Moreover, $\mathrm{Arg}_3(d_1 \to d_2; \ell) = \frac{5\beta}{7}$. This gives Table 4.1. Note that there are 12 choices of c' so we break the table into upper (for $c' \equiv 1, 5, 11, 13, 17, 19 \pmod 7$) and lower (for $c' \equiv 23, 25, 29, 31, 37, 41 \pmod 7$) parts.

Next we consider $\mathrm{Arg}(d_2 \to d_3; \ell)$, with $d_2 a_4 \equiv d_3 a_5 \equiv 1 \pmod 7$. We have

$$\mathrm{Arg}_1(d_2 \to d_3; \ell) = -\frac{3\beta c'^2\ell^2}{14} \begin{cases} +0 & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7})/\sin(\frac{\pi a_4 \ell}{7})) = 1, \\ \pm\frac{1}{2} & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_5 \ell}{7})/\sin(\frac{\pi a_4 \ell}{7})) = -1. \end{cases} \tag{4.6}$$

When $\ell = 1$, the sign changes when $\beta c' \equiv 8 \pmod{14}$. When $\ell = 2$, the sign remains the same. when $\ell = 3$, the sign changes when $3\beta c' \equiv 3 \pmod{14}$ but remains when $10 \pmod{14}$ because $a_4 \ell \equiv 5 \pmod 7$.

Since $12cs(d,c) \equiv 0 \pmod 6$, we have

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv -d_3 - a_5 + d_2 + a_4 \equiv -2\beta c' \pmod c,$$

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) - 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) \equiv 4 \pmod 8,$$

c' (mod 42)	1	5	11	13	17	19
β	1	3	2	6	5	3
$3\beta c'$ (mod 14)	3	3	10	10	3	3
$-9\beta c'^2$ (mod 14)	5	11	6	2	1	11
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{5}{14}$	$\frac{11}{14}$	$\frac{6}{14}+\frac{1}{2}$	$\frac{2}{14}+\frac{1}{2}$	$\frac{1}{14}$	$\frac{11}{14}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?	$+$			$-$		
$-18\beta c'^2 \equiv 3c'$ (mod 7)	3	1	5	4	2	1
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2}+\frac{3c'}{7}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 2)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$-\frac{3}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$		$-$	
$9\beta c'$ (mod 14)	9	9	2	2	9	9
$-81\beta c'^2$ (mod 14)	3	1	12	4	9	1
$\mathrm{Arg}_1(d_1 \to d_2; 3)$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 3)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$				$+$
c' (mod 42)	23	25	29	31	37	41
β	4	2	1	5	4	6
$3\beta c'$ (mod 14)	10	10	3	3	10	10
$-9\beta c'^2$ (mod 14)	10	6	5	1	10	2
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 1)$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$-\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$
$c'\ell = \pm 1$ (mod 7)?			$+$			$-$
$-18\beta c'^2 \equiv 3c'$ (mod 7)	6	5	3	2	8	4
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2}+\frac{3c'}{7}$	$\frac{5}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 1)$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$-\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$		$-$		
$9\beta c'$ (mod 14)	2	2	9	9	2	2
$-81\beta c'^2$ (mod 14)	6	12	3	9	6	4
$\mathrm{Arg}_1(d_1 \to d_2; 3)$	$\frac{3}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
$(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1 \to d_2; 1)$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$				$-$	

TABLE 4.1. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

and $-84s(d_3,c)+84s(d_2,c) \pmod{168}$ is uniquely determined by $12 \pmod{24}$ and $-2\beta \pmod 7$. So

$$\mathrm{Arg}_2(d_2\to d_3;\ell)=\frac{1,3,5,9,11,13}{14}\quad\text{when } \beta=1,3,5,2,4,6,\ \text{resp.}$$

Moreover, $\mathrm{Arg}_3(d_2\to d_3;\ell)=\frac{5\beta}{7}$. This gives Table 4.2, which is broken into upper (for $c'\equiv 1,5,11,13,17,19 \pmod{7})$) and lower (for $c'\equiv 23,25,29,31,37,41 \pmod{7})$) parts.

Then we investigate $\mathrm{Arg}(d_3\to d_4;\ell)$ with $d_3a_5\equiv d_4a_2\equiv 1 \pmod 7$. We have

$$\mathrm{Arg}_1(d_3\to d_4;\ell)=\frac{9\beta c'^2\ell^2}{14}\begin{cases}+0 & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_2\ell}{7})/\sin(\frac{\pi a_5\ell}{7}))=1,\\ \pm\frac12 & \text{if } \mathrm{sgn}(\sin(\frac{\pi a_2\ell}{7})/\sin(\frac{\pi a_5\ell}{7}))=-1.\end{cases} \tag{4.7}$$

When $\ell=1$, the sign changes if $3\beta c'\equiv 10 \pmod{14}$. When $\ell=2$, the sign always changes. When $\ell=3$, the sign changes if $9\beta c'\equiv 2 \pmod{14}$ but remains if $9\beta c'\equiv 9 \pmod{14}$.

We have $12cs(d,c)\equiv 0 \pmod 6$,

$$-12cs(d_4,c)+12cs(d_3,c)\equiv 2\beta c' \pmod c,$$

and

$$-12cs(d_4,c)+12cs(d_3,c)\equiv 2(\tfrac{d_4}{7})(\tfrac{d_4}{c'})-2(\tfrac{d_3}{7})(\tfrac{d_3}{c'})\equiv 4 \pmod 8.$$

So $-84s(d_4,c)+84s(d_3,c) \pmod{168}$ is uniquely determined by $12 \pmod{24}$ and $2\beta \pmod 7$ and

$$\mathrm{Arg}_2(d_3\to d_4;\ell)=\frac{1,3,5,9,11,13}{14}\quad\text{when } \beta=6,4,2,5,3,1,\ \text{resp.}$$

Moreover, $\mathrm{Arg}_3(d_3\to d_4;\ell)=\frac{5\beta}{7}$. This gives Table 4.3.

Now we have finished the proof of Condition 4.2 when $2\nmid c'$, $3\nmid c'$ and $7\nmid c'$ by comparing Table 4.1, Table 4.2, and Table 4.3.

4.2. **Case** $2\nmid c', 3|c', 7\nmid c'$**.** In this case $c'\equiv 3,9,15,27,33,39 \pmod{42}$. First we check $\mathrm{Arg}(d_1\to d_2;\ell)$ with $d_1a_1\equiv d_2a_4\equiv 1 \pmod 7$. For $\mathrm{Arg}_1(d_1\to d_2;\ell)$, we use (4.5). For Arg_2, we have $\theta=3$, $6cs(d,c)\in\mathbb{Z}$, and

$$-12cs(d_2,c)+12cs(d_1,c)\equiv -d_2-\overline{d_{2\{3c\}}}+d_1+\overline{d_{1\{3c\}}}\equiv -\beta c'+\beta c'\overline{d_{1\{3c\}}}\cdot\overline{d_{2\{3c\}}} \pmod{3c}.$$

Here $\overline{d_{1\{3c\}}}$ is the inverse of $d_1 \pmod{3c}$ and we have used (3.12). Hence we confirm that $-12cs(d_2,c)+12cs(d_1,c)$ is a multiple of c'. After dividing the above congruence by c', we obtain a congruence modulo 21 while $\overline{d_{j\{3c\}}}\equiv a_{\overline{j\{7\}}} \pmod{21}$ due to $21|c$. Hence

$$-84s(d_2,c)+84s(d_1,c)\equiv -\beta+\beta a_1a_4\equiv \beta(a_1a_4-1) \pmod{21}.$$

We have $a_1a_4\equiv 4 \pmod{21}$ by $a_4a_1\equiv 1 \pmod 3$ and $a_1a_4\equiv 4 \pmod 7$. Hence

$$-28s(d_2,c)+28s(d_1,c)\equiv \beta \pmod 7.$$

Due to $(\frac{2}{7})=1$, we also have

$$-12cs(d_1,c)+12cs(d_2,c)\equiv 2(\tfrac{d_1}{7})(\tfrac{d_1}{c'})-2(\tfrac{d_2}{7})(\tfrac{d_2}{c'})\equiv 0 \pmod 8.$$

Since $3c'$ is odd, we still have $-28s(d_2,c)+28s(d_1,c)\equiv 0 \pmod 8$. Now we get $\mathrm{Arg}_2(d_1\to d_2;\ell)=\frac{\overline{8_{\{21\}}}\cdot\beta}{7}=\frac{\beta}{7}$ and $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1\to d_2;\ell)=-\frac{\beta}{7}$. This gives Table 4.4.

Next we investigate $\mathrm{Arg}(d_2\to d_3;\ell)$ with $d_2a_4\equiv d_3a_5\equiv 1 \pmod 7$. For Arg_1 we use (4.6). For Arg_2, we have $\theta=3$, $6cs(d,c)\in\mathbb{Z}$, and

$$-12cs(d_3,c)+12cs(d_2,c)\equiv -d_3-\overline{d_{3\{3c\}}}+d_2+\overline{d_{2\{3c\}}}\equiv -\beta c'+\beta c'\overline{d_{2\{3c\}}}\cdot\overline{d_{3\{3c\}}} \pmod{3c}.$$

c' (mod 42)	1	5	11	13	17	19
β	1	3	2	6	5	3
$\beta c'$ (mod 14)	1	1	8	8	1	1
$-3\beta c'^2$ (mod 14)	11	13	2	10	5	13
$\mathrm{Arg}_1(d_2 \to d_3; 1)$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{2}{14} + \frac{1}{2}$	$\frac{10}{14} + \frac{1}{2}$	$\frac{5}{14}$	$\frac{13}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?	$+$			$-$		
$-6\beta c'^2 \equiv c'$ (mod 7)	1	5	4	6	3	5
$\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{3}{7}$	$\frac{5}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$		$-$	
$3\beta c'$ (mod 14)	3	3	10	10	3	3
$-27\beta c'^2$ (mod 14)	1	5	4	6	3	5
$\mathrm{Arg}_1(d_2 \to d_3; 3)$	$-\frac{3}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$				$+$
c' (mod 42)	23	25	29	31	37	41
β	4	2	1	5	4	6
$\beta c'$ (mod 14)	8	8	1	1	8	8
$-3\beta c'^2$ (mod 14)	8	2	11	5	8	10
$\mathrm{Arg}_1(d_2 \to d_3; 1)$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$			$-$
$-6\beta c'^2 \equiv c'$ (mod 7)	2	4	1	3	2	6
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{c'}{7}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{6}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$-\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$		$-$		
$3\beta c'$ (mod 14)	10	10	3	3	10	10
$-27\beta c'^2$ (mod 14)	2	4	1	3	2	6
$\mathrm{Arg}_1(d_2 \to d_3; 3)$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{3}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$				$-$	

TABLE 4.2. Table for $\mathrm{Arg}(d_2 \to d_3; \ell)$; $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

c' (mod 42)	1	5	11	13	17	19
β	1	3	2	6	5	3
$3\beta c'$ (mod 14)	3	3	10	10	3	3
$9\beta c'^2$ (mod 14)	9	3	8	12	13	3
$\mathrm{Arg}_1(d_3 \to d_4; 1)$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{8}{14} + \frac{1}{2}$	$\frac{12}{14} + \frac{1}{2}$	$\frac{13}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 1)$	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?	$+$			$-$		
$18\beta c'^2 \equiv 4c' \pmod 7$	4	6	2	3	5	6
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$	$\frac{1}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 2)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 2)$	$-\frac{2}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$		$-$	
$9\beta c'$ (mod 14)	9	9	2	2	9	9
$81\beta c'^2$ (mod 14)	11	13	2	10	5	13
$\mathrm{Arg}_1(d_3 \to d_4; 3)$	$\frac{11}{14}$	$\frac{13}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{13}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 3)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 3)$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$+$				$+$
c' (mod 42)	23	25	29	31	37	41
β	4	2	1	5	4	6
$3\beta c'$ (mod 14)	10	10	3	3	10	10
$9\beta c'^2$ (mod 14)	4	8	9	13	4	12
$\mathrm{Arg}_1(d_3 \to d_4; 1)$	$-\frac{3}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 1)$	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$			$-$
$18\beta c'^2 \equiv 4c' \pmod 7$	1	2	4	5	1	3
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{3c'}{7}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 2)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 2)$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$+$		$-$		
$9\beta c'$ (mod 14)	2	2	9	9	2	2
$-81\beta c'^2$ (mod 14)	8	2	11	5	8	10
$\mathrm{Arg}_1(d_3 \to d_4; 3)$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 3)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 3)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?	$-$				$-$	

TABLE 4.3. Table for $\mathrm{Arg}(d_3 \to d_4; \ell)$; $2 \nmid c$, $3 \nmid c$, $7 \nmid c$.

c' (mod 42)	3	9	15	27	33	39
β	5	4	1	6	3	2
$3\beta c'$ (mod 14)	3	10	3	10	3	10
$-9\beta c'^2$ (mod 14)	1	10	5	2	11	6
$\mathrm{Arg}_1(d_1 \to d_2; 1)$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$\frac{11}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$	$-$		
$-18\beta c'^2 \equiv 3c'$ (mod 7)	2	6	3	4	1	5
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$	$-\frac{3}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$\frac{1}{14}$	$-\frac{3}{14}$	$-\frac{3}{14}$	$\frac{3}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$					$+$
$9\beta c'$ (mod 14)	9	2	9	2	9	2
$-81\beta c'^2$ (mod 14)	9	6	3	4	1	12
$\mathrm{Arg}_1(d_1 \to d_2; 3)$	$\frac{1}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$\frac{3}{7}$	$-\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?		$-$			$+$	

TABLE 4.4. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2 \nmid c$, $3|c$, $7 \nmid c$.

Hence we confirm that $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of c'. After dividing by c', we obtain a congruence modulo 21 and

$$-84s(d_3, c) + 84s(d_2, c) \equiv -\beta + \beta a_4 a_5 \equiv \beta(a_4 a_5 - 1) \pmod{21}.$$

Since $a_4 a_5 \equiv 13 \pmod{21}$ by $a_4 a_5 \equiv 1 \pmod 3$ and $a_4 a_5 \equiv -1 \pmod 7$, we have

$$-28s(d_3, c) + 28s(d_2, c) \equiv 4\beta \pmod 7.$$

By (2.5), we get

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 2(\tfrac{d_2}{7})(\tfrac{d_2}{c'}) - 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) \equiv 4 \pmod 8.$$

Since $3c'$ is odd, we still have $-28s(d_3, c) + 28s(d_2, c) \equiv 4 \pmod 8$. Now $4\beta \pmod 7$ and $4 \pmod 8$ determines a unique residue modulo 56 and then

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) \equiv \frac{1, 3, 5, 9, 11, 13}{14} \pmod 1 \quad \text{when} \quad \beta = 1, 3, 5, 2, 4, 6.$$

This gives Table 4.5.

Finally we deal with $\mathrm{Arg}(d_3 \to d_4; \ell)$ where $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$. For Arg_1 we apply (4.7). For Arg_2, we have $\theta = 3$, $6cs(d, c) \in \mathbb{Z}$, and

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv -d_4 - \overline{d_{4\{3c\}}} + d_3 + \overline{d_{3\{3c\}}} \equiv -\beta c' + \beta c' \overline{d_{3\{3c\}}} \cdot \overline{d_{4\{3c\}}} \pmod{3c}.$$

c' (mod 42)	3	9	15	27	33	39
β	5	4	1	6	3	2
$\beta c'$ (mod 14)	1	8	1	8	1	8
$-3\beta c'^2$ (mod 14)	5	8	11	10	13	2
$\mathrm{Arg}_1(d_2 \to d_3; 1)$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?			+	−		
$-6\beta c'^2 \equiv c'$ (mod 7)	3	2	1	6	5	4
$\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{4}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$\frac{5}{14}$	$-\frac{1}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?	−					+
$3\beta c'$ (mod 14)	3	10	3	10	3	10
$-27\beta c'^2$ (mod 14)	3	2	1	6	5	4
$\mathrm{Arg}_1(d_1 \to d_2; 3)$	$-\frac{2}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		−			+	

TABLE 4.5. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2 \nmid c$, $3|c$, $7 \nmid c$.

We again confirm that $-12cs(d_4, c) + 12cs(d_3, c)$ is a multiple of c'. After dividing the above congruence by c', we obtain a congruence modulo 21 and

$$-84s(d_4, c) + 84s(d_3, c) \equiv -\beta + \beta a_5 a_2 \equiv \beta(a_5 a_2 - 1) \pmod{21}.$$

Since $a_2 a_5 \equiv 10 \pmod{21}$, we get

$$-28s(d_4, c) + 28s(d_3, c) \equiv 3\beta \pmod{7}.$$

We also have

$$-12cs(d_4, c) + 12cs(d_3, c) \equiv 2(\tfrac{d_4}{7})(\tfrac{d_4}{c'}) - 2(\tfrac{d_3}{7})(\tfrac{d_3}{c'}) \equiv 4 \pmod{8}.$$

Since $3c'$ is odd, we get $-28s(d_4, c) + 28s(d_3, c) \equiv 4 \pmod{8}$. Now $3\beta \pmod{7}$ and $4 \pmod{8}$ determines a unique residue modulo 56 and then

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) \equiv \frac{1, 3, 5, 9, 11, 13}{14} \pmod{1} \quad \text{when} \quad \beta = 6, 4, 2, 5, 3, 1.$$

This gives Table 4.6 and we have finished the proof for $c' \equiv 3, 9, 15, 27, 33, 39 \pmod{42}$.

4.3. **Case** $2|c', 3 \nmid c', 7 \nmid c'$**.** In this case $c' \equiv 2, 4, 8, 10, 16, 20, 22, 26, 32, 34, 38, 40 \pmod{42}$. We compute $\mathrm{Arg}(d_1 \to d_2; \ell)$ via (4.5). For Arg_2 we need to combine (2.3) and (2.5). We have $12cs(d, c) \equiv 0 \pmod{6}$ and

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv -d_2 - a_4 + d_1 + a_1 \equiv -4\beta c' \pmod{c}. \tag{4.8}$$

Then $-12cs(d_2, c) + 12cs(d_1, c)$ is a multiple of c'.

c' (mod 42)	3	9	15	27	33	39
β	5	4	1	6	3	2
$3\beta c'$ (mod 14)	3	10	3	10	3	10
$9\beta c'^2$ (mod 14)	13	4	9	12	3	8
$\mathrm{Arg}_1(d_3 \to d_4; 1)$	$\frac{13}{14}$	$-\frac{3}{14}$	$\frac{9}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$	$-$		
$18\beta c'^2 \equiv 4c' \pmod 7$	5	1	4	3	6	2
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$	$\frac{3}{14}$	$-\frac{5}{14}$	$\frac{1}{14}$	$-\frac{1}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 2)$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$\frac{2}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?	$-$					$+$
$9\beta c'$ (mod 14)	9	2	9	2	9	2
$81\beta c'^2$ (mod 14)	5	8	11	10	13	2
$\mathrm{Arg}_1(d_1 \to d_2; 3)$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{3}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$-$			$+$	

TABLE 4.6. Table for $\mathrm{Arg}(d_3 \to d_4; \ell)$; $2 \nmid c$, $3|c$, $7 \nmid c$.

We claim that

$$- 12cs(d_2, c) + 12cs(d_1, c) \equiv 0 \pmod{8 \times 2^\lambda}. \tag{4.9}$$

Denote $\lambda \geq 1$ by $2^\lambda \| c$. We have

$$\begin{aligned} -12cs(d_2, c) + 12cs(d_1, c) \equiv & - d_2 - \overline{d_{2\{8\times 2^\lambda\}}}(c^2 + 3c + 1 + 2c(\tfrac{c}{d_2})) \\ & + d_1 + \overline{d_{1\{8\times 2^\lambda\}}}(c^2 + 3c + 1 + 2c(\tfrac{c}{d_1})) \\ \equiv & - \beta c' + \beta c' \overline{d_{2\{8\times 2^\lambda\}}} \cdot \overline{d_{1\{8\times 2^\lambda\}}}(c^2 + 3c + 1) \\ & + 2c(\overline{d_{1\{8\times 2^\lambda\}}}(\tfrac{c}{d_1}) - \overline{d_{2\{8\times 2^\lambda\}}}(\tfrac{c}{d_2})) \pmod{8 \times 2^\lambda}. \end{aligned}$$

After dividing c', we get the value modulo 8 by $\overline{x_{\{8\}}} \equiv x \pmod 8$:

$$\begin{aligned} -84s(d_2, c) + 84s(d_1, c) & \equiv -\beta + \beta d_2 d_1 (c^2 + 3c + 1) + 6(d_1(\tfrac{c}{d_1}) - d_2(\tfrac{c}{d_2})) \\ & \equiv \beta c'(1 + d_1\beta)(c' + 1) - 2(d_1(\tfrac{c}{d_1}) - d_2(\tfrac{c}{d_2})) \pmod 8 \end{aligned}$$

For the first value val:$= \beta c'(1 + d_1\beta)(c' + 1) \pmod 8$, we see that both $\beta(1 + d_1\beta)$ and c' are even, hence the result is $0, 4 \pmod 8$. Moreover, val. is the same for c' and $c' + 7$. Then we have Table 4.7.

For the other part we determine whether

$$d_1(\tfrac{c}{d_1}) - d_2(\tfrac{c}{d_2}) \equiv 0 \text{ or } 2 \pmod 4. \tag{4.10}$$

c' (mod 7)	1	2	3	4	5	6
β	1	4	5	2	3	6
$\beta c'$	c'	$4c'$	$5c'$	$2c'$	$3c'$	$6c'$
βd_1+1	d_1+1	$4d_1+1$	$5d_1+1$	$2d_1+1$	$3d_1+1$	$6d_1+1$
$2\|c$, $d_1\equiv 1 \pmod 4$	4	0	4	4	0	4
$2\|c$, $d_1\equiv 3 \pmod 4$	0	0	0	4	4	4
$4\|c$	0	0	0	0	0	0

Table 4.7. Table for val:= $\beta c'\,(\beta d_1+1)\,(c'+1) \pmod 8$; $2|c$, no requirement for $(c,3)$, $7\nmid c$.

When $4|c$, (4.10) is always $0 \pmod 4$, which proves (4.9) by combining the last row of Table 4.7.

When $2\|c$, by $(\frac{7}{x})=(\frac{x}{7})(-1)^{\frac{x-1}{2}}$ for odd x, we have

$$d_1(\tfrac{c}{d_1})-d_2(\tfrac{c}{d_2})\equiv(\tfrac{d_1}{c'/2})\left((-1)^{\frac{d_1-1}{2}+\frac{d_1-1}{2}\cdot\frac{\frac{c'}{2}-1}{2}}(\tfrac{2}{d_1})d_1-(-1)^{\frac{d_2-1}{2}+\frac{d_2-1}{2}\cdot\frac{\frac{c'}{2}-1}{2}}(\tfrac{2}{d_2})d_2\right) \pmod 4. \tag{4.11}$$

Since $d_2=d_1+\beta c'$, we divide into cases for $c'\equiv 2,6 \pmod 8$, $d_1\equiv 1,3,5,7 \pmod 8$ and β from 1 to 6 to make Table 4.8. Note that $d_2 \pmod 8$ is derived by $c' \pmod 8$, β and $d_1 \pmod 8$.

(4.10) ↘	$c'\equiv 2 \pmod 8$				$c'\equiv 6 \pmod 8$			
$d_1 \pmod 8$	1	3	5	7	1	3	5	7
$\beta=1$	2	0	2	0	2	0	2	0
$\beta=4$	0	0	0	0	0	0	0	0
$\beta=5$	2	0	2	0	2	0	2	0
$\beta=2$	2	2	2	2	2	2	2	2
$\beta=3$	0	2	0	2	0	2	0	2
$\beta=6$	2	2	2	2	2	2	2	2

Table 4.8. Table for (4.10); $2|c$, no requirement for $(c,3)$, $7\nmid c$.

Comparing Table 4.7 and Table 4.8, we have proved (4.9). Combining (4.8) and $12cs(d,c)\equiv 0 \pmod 6$, we divide $24c'$ to compute $\mathrm{Arg}_2(d_1\to d_2;\ell)=\frac{\beta}{7}$. Then $(\mathrm{Arg}_2+\mathrm{Arg}_3)(d_1\to d_2;\ell)=-\frac{\beta}{7}$ and we have Table 4.9.

Next we deal with $\mathrm{Arg}(d_2\to d_3;\ell)$ with $d_2a_4\equiv d_3a_5\equiv 1 \pmod 7$. For Arg_1 we apply (4.6). For $\mathrm{Arg}_2(d_2\to d_3;\ell)$ we do the similar proof as $\mathrm{Arg}_2(d_1\to d_2;\ell)$. First we have

$$-12cs(d_3,c)+12cs(d_2,c)\equiv -d_3-a_5+d_2+a_4\equiv -2\beta c' \pmod c. \tag{4.12}$$

Then $-12cs(d_3,c)+12cs(d_2,c)$ is a multiple of c'.

We claim that

$$-12cs(d_3,c)+12cs(d_2,c)\equiv 4\times 2^{\lambda} \pmod{8\times 2^{\lambda}}. \tag{4.13}$$

c' (mod 42)	2	4	8	10	16	20
β	4	2	1	5	4	6
$-3\beta c'^2$ (mod 14)	10	6	12	8	10	2
$\mathrm{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{4}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{6}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$-\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$			$-$
$-18\beta c'^2 \equiv 3c'$ (mod 7)	6	5	3	2	6	4
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$	$\frac{5}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{6}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$-\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$		$-$		
$-81\beta c'^2$ (mod 14)	6	12	10	2	6	4
$\mathrm{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$	$\frac{3}{7}$	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{4}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{6}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$				$-$	

c' (mod 42)	22	26	32	34	38	40
β	1	3	2	6	5	3
$-9\beta c'^2$ (mod 14)	12	4	6	2	8	4
$\mathrm{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{6}{7}$	$-\frac{5}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{5}{14}$
$c'\ell = \pm 1$ (mod 7)?	$+$			$-$		
$-18\beta c'^2 \equiv 3c'$ (mod 7)	3	1	5	4	2	1
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{6}{7}$	$-\frac{5}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$-\frac{3}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$		$-$	
$-81\beta c'^2$ (mod 14)	10	8	12	4	2	8
$\mathrm{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{6}{7}$	$-\frac{5}{7}$	$-\frac{3}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$\frac{1}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$				$+$

TABLE 4.9. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3 \nmid c$, $7 \nmid c$.

For $\lambda \geq 1$ such that $2^\lambda \| c$, we have

$$\begin{aligned}-12cs(d_3,c)+12cs(d_2,c) \equiv & -d_3-\overline{d_{3\{8\times 2^\lambda\}}}(c^2+3c+1+2c(\tfrac{c}{d_3}))\\ & +d_2+\overline{d_{2\{8\times 2^\lambda\}}}(c^2+3c+1+2c(\tfrac{c}{d_2}))\\ \equiv & -\beta c'+\beta c'\overline{d_{3\{8\times 2^\lambda\}}}\cdot\overline{d_{2\{8\times 2^\lambda\}}}(c^2+3c+1)\\ & +2c(\overline{d_{2\{8\times 2^\lambda\}}}(\tfrac{c}{d_2})-\overline{d_{3\{8\times 2^\lambda\}}}(\tfrac{c}{d_3}))\ (\mathrm{mod}\ 8\times 2^\lambda),\end{aligned}$$

After dividing c', since $2^\lambda \| c'$ and $\overline{x_{\{8\}}} \equiv x \pmod 8$ for odd x, we have

$$\begin{aligned}-84s(d_3,c)+84s(d_2,c) &\equiv -\beta+\beta d_3 d_2(c^2+3c+1)+6(d_2(\tfrac{c}{d_2})-d_3(\tfrac{c}{d_3}))\\ &\equiv \beta c'(1+d_2\beta)(c'+1)-2(d_2(\tfrac{c}{d_2})-d_3(\tfrac{c}{d_3}))\ (\mathrm{mod}\ 8)\end{aligned}$$

The proof of (4.13) is then the same as the proof of (4.9) before, noting that in the second part we have $(\frac{d_2}{7}) = 1$ while $(\frac{d_3}{7}) = -1$. This difference makes an alternation in Table 4.8 where we should change all 2 to 0 and all 0 to 2, which results in $4 \times 2^\lambda \pmod{8 \times 2^\lambda}$ rather than $0 \pmod{8 \times 2^\lambda}$ in (4.13). We omit the details.

Combining (4.12), (4.13) and $12cs(d,c) \equiv 0 \pmod 6$ we can determine $\mathrm{Arg}_2(d_2 \to d_3;\ell)$ with denominator 42 and numerator by $3\beta \pmod 7$ and $3 \pmod 6$, hence

$$\mathrm{Arg}_2(d_2 \to d_3;\ell) = \frac{1,3,5,9,11,13}{14} \quad \text{when } \beta = 1,3,5,2,4,6, \text{ resp.}$$

Now we have Table 4.10.

Finally we check $\mathrm{Arg}(d_3 \to d_4;\ell)$ with $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$. For Arg_1 we apply (4.7). Since c' is even, we have $-3\beta c' \equiv 4 \pmod{14}$ and the sign always changes.

For $\mathrm{Arg}_2(d_3 \to d_4;\ell)$, first we have

$$-12cs(d_4,c)+12cs(d_3,c) \equiv -d_4-a_2+d_3+a_5 \equiv 2\beta c' \pmod c. \tag{4.14}$$

Then $-12cs(d_4,c)+12cs(d_3,c)$ is a multiple of c'. We claim that

$$-12cs(d_4,c)+12cs(d_3,c) \equiv 4\times 2^\lambda \pmod{8\times 2^\lambda}. \tag{4.15}$$

The proof is the same as the proof for (4.13) and we omit the details. Combining (4.14), (4.15) and (2.2), we can determine $\mathrm{Arg}_2(d_3 \to d_4;\ell)$ with denominator 42 and numerator by $4\beta \pmod 7$ and $3 \pmod 6$, hence

$$\mathrm{Arg}_2(d_2 \to d_3;\ell) = \frac{1,3,5,9,11,13}{14} \quad \text{when } \beta = 6,4,2,5,3,1.$$

This gives Table 4.11.

Comparing Tables 4.9, 4.10 and 4.11 we see that when $2|c'$, $3 \nmid c'$ and $7 \nmid c'$, Condition 4.2 holds and we have proved Proposition 4.1 in this case.

4.4. **Case** $2|c', 3|c', 7 \nmid c'$**.** In this case $c' \equiv 6,12,18,24,30,36 \pmod{42}$. We deal with $\mathrm{Arg}(d_1 \to d_2;\ell)$ by (4.5). For Arg_2 we need to combine (2.2) and (2.5). We have $12cs(d,c) \equiv 0 \pmod 2$ and

$$-12cs(d_2,c)+12cs(d_1,c) \equiv -d_2-\overline{d_{2\{3c\}}}+d_1+\overline{d_{1\{3c\}}} \equiv -\beta c'+\beta c'\overline{d_{2\{3c\}}}\cdot\overline{d_{1\{3c\}}} \pmod{3c}. \tag{4.16}$$

Then $-12cs(d_2,c)+12cs(d_1,c)$ is a multiple of c'. After dividing c', since $3|c'$ and $\overline{d_{j\{3c\}}} \equiv a_{\overline{j\{7\}}} \pmod{21}$, we get

$$-84s(d_2,c)+84s(d_1,c) \equiv -\beta+\beta a_4 a_1 \equiv 3\beta \pmod{21}. \tag{4.17}$$

c' (mod 42)	2	4	8	10	16	20
β	4	2	1	5	4	6
$-3\beta c'^2$ (mod 14)	8	2	4	12	8	10
$\mathrm{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$			$-$
$-6\beta c'^2 \equiv c'$ (mod 7)	2	4	1	3	2	6
$\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{3c'}{7}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{6}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$-\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$		$-$		
$-27\beta c'^2$ (mod 14)	2	4	8	10	2	6
$\mathrm{Arg}_1(d_2 \to d_3; 3) : -\frac{27\beta c'^2}{14}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{3}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$	$\frac{9}{14}$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{3}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$				$-$	

c' (mod 42)	22	26	32	34	38	40
β	1	3	2	6	5	3
$-3\beta c'^2$ (mod 14)	4	6	2	10	12	6
$\mathrm{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{2}{7}$
$c'\ell = \pm 1$ (mod 7)?	$+$			$-$		
$-6\beta c'^2 \equiv c'$ (mod 7)	1	5	4	6	3	5
$\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{3c'}{7}$	$\frac{1}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{3}{7}$	$\frac{5}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$		$-$	
$-27\beta c'^2$ (mod 14)	8	12	4	6	10	12
$\mathrm{Arg}_1(d_2 \to d_3; 3) : -\frac{27\beta c'^2}{14}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{5}{7}$	$\frac{6}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{13}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$				$+$

TABLE 4.10. Table for $\mathrm{Arg}(d_2 \to d_3; \ell)$; $2|c$, $3 \nmid c$, $7 \nmid c$.

c' (mod 42)	2	4	8	10	16	20
β	4	2	1	5	4	6
$9\beta c'^2$ (mod 14)	4	8	2	6	4	12
$\mathrm{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$	$-\frac{3}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$			$-$
$18\beta c'^2 \equiv 4c' \pmod 7$	1	2	4	5	1	3
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$-\frac{5}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$+$		$-$		
$81\beta c'^2$ (mod 14)	8	2	4	12	8	10
$\mathrm{Arg}_1(d_3 \to d_4; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{1}{14}$	$\frac{11}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 3)$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?	$-$				$-$	

c' (mod 42)	22	26	32	34	38	40
β	1	3	2	6	5	3
$9\beta c'^2$ (mod 14)	2	10	8	12	6	10
$\mathrm{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 1)$	$\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$c'\ell = \pm 1 \pmod 7$?	$+$			$-$		
$18\beta c'^2 \equiv 4c' \pmod 7$	4	6	2	3	5	6
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$	$\frac{1}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 2)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$-\frac{2}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$		$-$	
$81\beta c'^2$ (mod 14)	4	6	2	10	12	6
$\mathrm{Arg}_1(d_2 \to d_3; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{9}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{13}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$+$				$+$

TABLE 4.11. Table for $\mathrm{Arg}(d_3 \to d_4; \ell)$; $2|c$, $3 \nmid c$, $7 \nmid c$.

where the last congruence equality follows by $a_4a_1 \equiv 4 \pmod 7$ and $1 \pmod 3$.

We still have (4.9) in this case:

$$-12cs(d_2, c) + 12cs(d_1, c) \equiv 0 \pmod{8 \times 2^\lambda}$$

because the proof of (4.9) does not depend on whether $3|c$ or not. Combining the above two congruences we have $\mathrm{Arg}_2(d_1 \to d_2; \ell) = \frac{\beta}{7}$. Then $(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; \ell) = -\frac{\beta}{7}$, which gives Table 4.12.

$c' \pmod{42}$	6	12	18	24	30	36
β	6	3	2	5	4	1
$-9\beta c'^2 \pmod{14}$	2	4	6	8	10	12
$\mathrm{Arg}_1(d_1 \to d_2; 1) : \frac{1}{2} - \frac{9\beta c'^2}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$	$\frac{3}{14}$	$\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 1)$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 1)$	$-\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$
$c'\ell \equiv \pm 1 \pmod 7$?	$-$					$+$
$-18\beta c'^2 \equiv 3c' \pmod 7$	4	1	5	2	6	3
$\mathrm{Arg}_1(d_1 \to d_2; 2) : \frac{1}{2} + \frac{3c'}{7}$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{3}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 2)$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 2)$	$\frac{3}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$	$-\frac{3}{14}$
$c'\ell \equiv \pm 1 \pmod 7$?			$+$	$-$		
$-81\beta c'^2 \pmod{14}$	4	8	12	2	6	10
$\mathrm{Arg}_1(d_1 \to d_2; 3) : -\frac{81\beta c'^2}{14}$	$\frac{2}{7}$	$\frac{4}{7}$	$\frac{6}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{5}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_1 \to d_2; 2)$	$-\frac{6}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{5}{7}$	$-\frac{4}{7}$	$-\frac{1}{7}$
Total $\mathrm{Arg}(d_1 \to d_2; 3)$	$\frac{3}{7}$	$\frac{1}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{1}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1 \pmod 7$?		$+$			$-$	

TABLE 4.12. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3|c$, $7 \nmid c$.

Next we check $\mathrm{Arg}(d_2 \to d_3; \ell)$ with $d_2a_4 \equiv d_3a_5 \equiv 1 \pmod 7$. We compute Arg_1 via (4.6). For $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ we have

$$-12cs(d_3, c)+12cs(d_2, c) \equiv -d_3-\overline{d_{3\{3c\}}}+d_2+\overline{d_{2\{3c\}}} \equiv -\beta c'+\beta c'\overline{d_{3\{3c\}}}\cdot\overline{d_{2\{3c\}}} \pmod c. \quad (4.18)$$

Then $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of c'. After dividing by c' we get

$$-84s(d_3, c) + 84s(d_2, c) \equiv -\beta + \beta a_5 a_4 \equiv 12\beta \pmod{21}. \quad (4.19)$$

The equality (4.13) still holds:

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}$$

because its proof does not involve whether $3|c'$ or not. Combining the two congruences above we can decide $\mathrm{Arg}_2(d_2 \to d_3; \ell)$ via $4\beta \pmod 7$ and $4 \pmod 8$:

$$\mathrm{Arg}_2(d_2 \to d_3; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 1, 3, 5, 2, 4, 6.$$

This gives Table 4.13.

c' (mod 42)	6	12	18	24	30	36
β	6	3	2	5	4	1
$-3\beta c'^2$ (mod 14)	10	6	2	12	8	4
$\mathrm{Arg}_1(d_2 \to d_3; 1) : \frac{1}{2} - \frac{3\beta c'^2}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 1)$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 1)$	$\frac{3}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$					$+$
$-6\beta c'^2 \equiv c'$ (mod 7)	6	5	4	3	2	1
$\mathrm{Arg}_1(d_2 \to d_3; 2) : \frac{c'}{7}$	$\frac{6}{7}$	$\frac{5}{7}$	$\frac{4}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{1}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 2)$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 2)$	$\frac{1}{14}$	$\frac{1}{14}$	$-\frac{5}{14}$	$\frac{5}{14}$	$-\frac{1}{14}$	$-\frac{1}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$	$-$		
$-27\beta c'^2$ (mod 14)	6	12	4	10	2	8
$\mathrm{Arg}_1(d_1 \to d_2; 3) : -\frac{27\beta c'^2}{14}$	$\frac{3}{7}$	$\frac{6}{7}$	$\frac{2}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{4}{7}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_2 \to d_3; 3)$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$\frac{13}{14}$	$\frac{9}{14}$	$\frac{11}{14}$
Total $\mathrm{Arg}(d_2 \to d_3; 3)$	$-\frac{5}{14}$	$\frac{3}{14}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{3}{14}$	$\frac{5}{14}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$			$-$	

TABLE 4.13. Table for $\mathrm{Arg}(d_1 \to d_2; \ell)$; $2|c$, $3|c$, $7 \nmid c$.

Finally we check $\mathrm{Arg}(d_3 \to d_4; \ell)$ with $d_3 a_5 \equiv d_4 a_2 \equiv 1 \pmod 7$. For Arg_1 we apply (4.7). Since c' is even, we have $-3\beta c' \equiv 4 \pmod{14}$ and the sign always changes.

For $\mathrm{Arg}_2(d_3 \to d_4; \ell)$, first we have

$$-12cs(d_4, c)+12cs(d_3, c) \equiv -d_4-\overline{d_{4\{3c\}}}+d_3+\overline{d_{3\{3c\}}} \equiv -\beta c'+\beta c'\overline{d_{4\{3c\}}}\cdot\overline{d_{3\{3c\}}} \pmod{3c}. \quad (4.20)$$

Then $-12cs(d_3, c) + 12cs(d_2, c)$ is a multiple of c'. After dividing c' we have

$$-84s(d_4, c) + 84s(d_3, c) \equiv -\beta + \beta a_2 a_5 \equiv 9\beta \pmod{21}.$$

We also have (4.15):

$$-12cs(d_3, c) + 12cs(d_2, c) \equiv 4 \times 2^\lambda \pmod{8 \times 2^\lambda}$$

because its proof does not involve whether $3|c'$ or not. Combining the two congruence equations above we can decide $\mathrm{Arg}_2(d_3 \to d_4; \ell)$ with denominator 56 and numerator determined by $3\beta \pmod 7$ and $4 \pmod 8$, hence

$$\mathrm{Arg}_2(d_3 \to d_4; \ell) = \frac{1, 3, 5, 9, 11, 13}{14} \quad \text{when } \beta = 6, 4, 2, 5, 3, 1.$$

This gives Table 4.14.

Comparing Tables 4.12, 4.13 and 4.14 we see that when $2|c'$, $3|c'$ and $7 \nmid c'$, Condition 4.2 holds.

We have proved Proposition 4.1 when $7\|c$. In the following subsection we prove Proposition 4.1 when $49|c$.

c' (mod 42)	6	12	18	24	30	36
β	6	3	2	5	4	1
$9\beta c'^2$ (mod 14)	12	10	8	6	4	2
$\mathrm{Arg}_1(d_3 \to d_4; 1) : \frac{1}{2} + \frac{9\beta c'^2}{14}$	$\frac{5}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$-\frac{1}{14}$	$-\frac{3}{14}$	$-\frac{5}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 1)$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 1)$	$-\frac{2}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{2}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?	$-$					$+$
$18\beta c'^2 \equiv 4c'$ (mod 7)	3	6	2	5	1	4
$\mathrm{Arg}_1(d_3 \to d_4; 2) : \frac{1}{2} + \frac{4c'}{7}$	$-\frac{1}{14}$	$\frac{5}{14}$	$-\frac{3}{14}$	$\frac{3}{14}$	$-\frac{5}{14}$	$\frac{1}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 2)$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 2)$	$\frac{2}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?			$+$	$-$		
$81\beta c'^2$ (mod 14)	10	6	2	12	8	4
$\mathrm{Arg}_1(d_3 \to d_4; 3) : \frac{1}{2} + \frac{81\beta c'^2}{14}$	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{5}{14}$	$\frac{5}{14}$	$\frac{1}{14}$	$-\frac{3}{14}$
$(\mathrm{Arg}_2 + \mathrm{Arg}_3)(d_3 \to d_4; 3)$	$\frac{5}{14}$	$\frac{13}{14}$	$\frac{11}{14}$	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{9}{14}$
Total $\mathrm{Arg}(d_3 \to d_4; 3)$	$-\frac{3}{7}$	$-\frac{1}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$\frac{1}{7}$	$\frac{3}{7}$
$c'\ell \equiv \pm 1$ (mod 7)?		$+$			$-$	

TABLE 4.14. Table for $\mathrm{Arg}(d_4 \to d_3; \ell)$; $2|c$, $3|c$, $7 \nmid c$.

4.5. **Case** $7|c'$**.** We still denote $c' = c/7$ and $V(r,c) = \{d \pmod c)^* : \ d \equiv r \pmod{c'}\}$ for $r \pmod{c'}^*$. Now $|V(r,c)| = 7$. Since $(d + c', c) = 1$ when $(d, c) = 1$, we can write $V(r,c) = \{d, d + c', d + 2c', \cdots, d + 6c'\}$ for $1 \le d < c'$ and $d \equiv r \pmod{c'}$.

We claim that Proposition 4.1 is still true:

$$s_{r,c} := \sum_{d \in V(r,c)} \frac{e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})} e\left(-\frac{12cs(d,c)}{24c}\right) e\left(\frac{5d}{c}\right) = 0, \tag{4.21}$$

while this time we have seven summands. We prove (4.21) by showing that there are only three cases for the sum: all at the outer circle (radius $1/\sin(\frac{\pi}{7})$), all at the middle circle (radius $1/\sin(\frac{2\pi}{7})$), and all at the inner circle (radius $1/\sin(\frac{3\pi}{7})$). Moreover, the seven points are equally distributed. Similar as before, we still denote P_1, P_2 and P_3 for each term in (4.21) and investigate the argument differences contributed from each term.

For any $d \in V(r,c)$ and $a \pmod c$ such that $ad \equiv 1 \pmod c$, we define $d_* = d + c'$ and a_* by $a_* d_* \equiv 1 \pmod c$. Specifically, we take $a_* = a - c'$, $a - 2c'$, $a + 3c'$, $a + 3c'$, $a - 2c'$, $a - c'$, when $d \equiv 1, 2, 3, 4, 5, 6 \pmod 7$, respectively. Note that $P_1(d) = (-1)^{ca\ell}/\sin(\frac{\pi a\ell}{7})$ has period c'. Hence we always have

$$\mathrm{Arg}_1(d \to d_*; \ell) = 0 \quad \text{and} \quad \mathrm{Arg}_3(d \to d_*; \ell) = \tfrac{5}{7}.$$

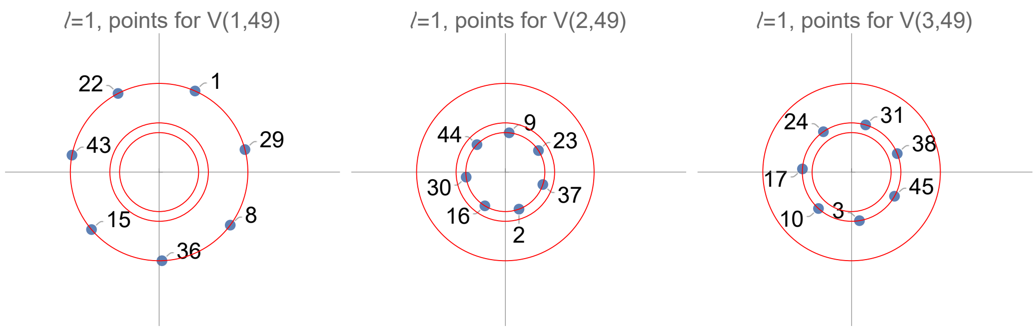
In the following two cases, we prove

$$\operatorname{Arg}(d \to d_*; \ell) = \begin{cases} -\frac{2}{7} & d \equiv 1, 6 \pmod 7; \\ \frac{3}{7} & d \equiv 2, 5 \pmod 7; \\ -\frac{1}{7} & d \equiv 3, 4 \pmod 7. \end{cases} \tag{4.22}$$

when $\ell = 1$. In the other cases $\ell = 2, 3$, only P_1 is affected and we still get (4.22).

Proof of Proposition 4.1 when $49|c$. It is clear that (4.22) implies (4.21). □

One may visualize (4.22) in the following graphs:



4.5.1. *c is odd.* First we suppose $3 \nmid c$. When $d \equiv 1, 6 \pmod 7$, by (2.2) we have $-12cs(d_*, c) + 12cs(d, c) \equiv 0 \pmod 6$,

$$-12cs(d_*, c) + 12cs(d, c) \equiv -d_* - a_* + d + a \equiv 0 \pmod c, \tag{4.23}$$

and by (2.4) we have

$$-12cs(d_*, c) + 12cs(d, c) \equiv 2\left(\tfrac{d_*}{c}\right) - 2\left(\tfrac{d}{c}\right) \equiv 0 \pmod 8$$

because $\left(\frac{d+c'}{c}\right) = \left(\frac{d+c'}{7}\right)\left(\frac{d+c'}{c'}\right) = \left(\frac{d}{7}\right)\left(\frac{d}{c'}\right) = \left(\frac{d}{c}\right)$ always. Then $-12cs(d_*, c) + 12cs(d, c) \equiv 0 \pmod{24c}$ and $\operatorname{Arg}_2(d \to d_*; \ell) = 0$. Since $\operatorname{Arg}_3(d \to d_*; \ell) = \frac{5}{7}$, we have proved (4.22).

When $d \equiv 2, 5 \pmod 7$, only (4.23) is affected and becomes

$$-12cs(d_*, c) + 12cs(d, c) \equiv -d_* - a_* + d + a \equiv c' \pmod c. \tag{4.24}$$

After dividing $24c'$ we get $\operatorname{Arg}_2(d \to d_*; \ell) = \frac{5}{7}$. We have proved (4.22) in this case.

When $d \equiv 3, 4 \pmod 7$, (4.23) becomes

$$-12cs(d_*, c) + 12cs(d, c) \equiv -d_* - a_* + d + a \equiv -4c' \pmod c. \tag{4.25}$$

We get $\operatorname{Arg}_2(d \to d_*; \ell) = \frac{1}{7}$ and (4.22).

Then we investigate the case $3|c'$. The following congruence

$$-12cs(d_*, c) + 12cs(d, c) \equiv 2\left(\tfrac{d_*}{c}\right) - 2\left(\tfrac{d}{c}\right) \equiv 0 \pmod 8$$

still holds and we compute

$$-12cs(d_*, c) + 12cs(d, c) \equiv -d_* - \overline{d_{1\{3c\}}} + d + \overline{d_{\{3c\}}} \equiv -c' + c'\overline{d_{1\{3c\}}} \cdot \overline{d_{\{3c\}}} \pmod{3c},$$

so

$$-84s(d_*, c) + 84s(d, c) \equiv -1 + a_* a \pmod{21}.$$

Since $a_*a \equiv 1 \pmod 3$ and $a_* \equiv a \pmod 7$, we have

$$-84s(d_*,c)+84s(d,c)\equiv\begin{cases}0 \pmod{21} & \text{if } d\equiv 1,6 \pmod 7,\\ 15 \pmod{21} & \text{if } d\equiv 2,5 \pmod 7,\\ 3 \pmod{21} & \text{if } d\equiv 3,4 \pmod 7.\end{cases} \tag{4.26}$$

Then $-28s(d_*,c)+28s(d,c)\equiv 0,5,1 \pmod 7$ and $\mathrm{Arg}_2(d\to d_*;\ell)=\frac{0,5,1}{7}$, respectively. We have proved (4.22) when c is odd.

4.5.2. *c is even.* The first case is $3\nmid c'$. Congruences (4.23), (4.24) and (4.25) are still valid here. By (2.5), we define $\lambda\geq 1$ by $2^\lambda \| c'$ and claim that

$$-12cs(d_*,c)+12cs(d,c)\equiv 0 \pmod{8\times 2^\lambda} \tag{4.27}$$

To compute this, we have

$$\begin{aligned}-12cs(d_*,c)+12cs(d,c)\equiv&-d_*-\overline{d_{1\{8\times 2^\lambda\}}}(c^2+3c+1)-\overline{d_{1\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d_*})\\ &+d+\overline{d_{\{8\times 2^\lambda\}}}(c^2+3c+1)+\overline{d_{\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d}) \pmod{8\times 2^\lambda}\\ \equiv&-c'+c'(c^2+3c+1)\overline{d_{1\{8\times 2^\lambda\}}}\cdot\overline{d_{\{8\times 2^\lambda\}}}\\ &-\overline{d_{1\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d_*})+\overline{d_{\{8\times 2^\lambda\}}}\cdot 2c(\tfrac{c}{d}) \pmod{8\times 2^\lambda}.\end{aligned}$$

After dividing c' we have

$$\begin{aligned}-84s(d_*,c)+84s(d,c)&\equiv -1+d_*d(c^2+3c+1)+2(\tfrac{c}{d_*})d_*-2(\tfrac{c}{d})d \pmod 8\\ &\equiv c'(c'+1)(d+1)+2(\tfrac{c}{d_*})d_*-2(\tfrac{c}{d})d \pmod 8.\end{aligned}$$

We also get

$$c'(c'+1)(d+1)\equiv\begin{cases}4 \pmod 8 & \text{if } 2\|c,\ d\equiv 1 \pmod 4,\\ 0 \pmod 8 & \text{if } 2\|c,\ d\equiv 3 \pmod 4,\\ 0 \pmod 8 & \text{if } 4|c.\end{cases} \tag{4.28}$$

When $4|c$, it is not hard to show $(\frac{c}{d_*})d_*-(\frac{c}{d})d\equiv 0 \pmod 4$ and we have proved (4.27) in this case.

When $2\|c$, we have Table 4.15 for val:$=(\frac{c}{d_*})d_*-(\frac{c}{d})d \pmod 4$ using quadratic reciprocity. Combining Table 4.15 and (4.28) we obtain (4.27).

$d \pmod 8$	1	3	5	7
$d_* \pmod 8$ when $c'\equiv 2 \pmod 8$	3	5	7	1
val	2	0	2	0
$d_* \pmod 8$ when $c'\equiv 6 \pmod 8$	7	1	3	5
val	2	0	2	0

TABLE 4.15. Table for val:$=(\frac{c}{d_*})d_*-(\frac{c}{d})d \pmod 4$; $2|c$, no requirement for $(3,c)$, $7|c$.

Combining (4.27) with (4.23), (4.24) and (4.25), we have proved (4.22) when $2|c$ and $3\nmid c$. When $3|c$, we use (4.26) instead of (4.23), (4.24) and (4.25). This finishes the proof of (4.22).

We have proved Proposition 4.1, which implies (7-5,1) of Theorem 1.3.

5. Proof of (7-5,2) of Theorem 1.3

For all $1\le \ell\le 6$, $n\ge 0$, $7|c$ and $7\nmid A$, if $A\ell\equiv \pm1 \pmod 7$ and $c=7A$, (7-5,2) becomes

$$e(\tfrac18)S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7)+2i\sqrt7 S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)=0. \tag{5.1}$$

We still denote $c'=c/7=A$ and $V(r,c):=\{d \pmod c)^*:\ d\equiv r \pmod c\}$ for $(r,c')=1$.

Recall (2.7) for $p=7$. By $\ell c\equiv \ell A \pmod 2$ we have

$$e(\tfrac18)S^{(\ell)}_{\infty\infty}(0,7n+5,c,\mu_7)=\sum_{d \pmod c)^*}\frac{(-1)^{\ell A}\exp\left(-\frac{3\pi i c' a\ell^2}{7}\right)}{\sin(\frac{\pi a\ell}{7})}e^{-\pi i s(d,c)}e\left(\frac{(7n+5)d}{c}\right). \tag{5.2}$$

Recall that $[A\ell]$ is defined as the least non-negative residue of $A\ell \pmod 7$. By (2.8), when $[A\ell]=1$, we denote T by $A\ell=7T+1$ and

$$\begin{aligned}&2i\sqrt7 S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)\\&=2i\sqrt7(-1)^{A\ell-[A\ell]}\sum_{\substack{B \pmod A)^*\\0<C<7A,7|C\\BC\equiv-1(A)}}e\left(\frac{(\frac32T^2+\frac12T)C}{A}\right)e^{-\pi is(B,A)}e\left(\frac{(7n+5)B}{A}\right).\end{aligned} \tag{5.3}$$

By (2.9), when $[A\ell]=6$, we denote T by $A\ell=7T-1$ and have

$$\begin{aligned}&2i\sqrt7 S^{(\ell)}_{0\infty}(0,7n+5,A,\mu_7;0)\\&=2i\sqrt7(-1)^{A\ell-[A\ell]}\sum_{\substack{B \pmod A)^*\\0<C<7A,7|C\\BC\equiv-1(A)}}e\left(\frac{(\frac32(T-1)^2+\frac52(T-1)+1)C}{A}\right)e^{-\pi is(B,A)}e\left(\frac{(7n+5)B}{A}\right).\end{aligned} \tag{5.4}$$

For $(r,c')=1$ and any $d\in V(r,c)$, we define $P(d)$ as

$$P(d):=\frac{(-1)^{[A\ell]}e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})}e^{-\pi is(d,c)}e\left(\frac{(7n+5)d}{c}\right)=:P_1(d)\cdot P_2(d)\cdot P_3(d). \tag{5.5}$$

When $A\ell=7T+1$, we denote $Q_1(B)=i$, $Q_3(B)=e(\frac{(7n+5)B}{A})$,

$$Q_2(B):=e\left(\frac{(\frac32T^2+\frac12T)C}{A}\right)e^{-\pi is(B,A)}\quad\text{and}\quad Q(B)=:2\sqrt7\cdot Q_1(B)Q_2(B)Q_3(B); \tag{5.6}$$

when $A\ell=7T-1$, we only change the definition of $Q_2(B)$ to

$$Q_2(B):=e\left(\frac{(\frac32(T-1)^2+\frac52(T-1)+1)C}{A}\right)e^{-\pi is(B,A)} \tag{5.7}$$

and still denote $Q(B)=2\sqrt7\cdot Q_1(B)Q_2(B)Q_3(B)$.

We divide the cases according to $c'\ell\equiv\pm1 \pmod 7$, ℓ, and the divisibility of A by $2,3$. For each $r \pmod A)^*$, recall that $d_1\in V(r,c)$ is the unique $d_1 \pmod c)^*$ such that $d_1\equiv 1 \pmod 7$. We compare the argument difference from $Q(B)$ to $P(d_1)$, where we choose

$$B=\begin{cases}-d_1T, & A\ell=7T+1,\\ d_1T, & A\ell=7T-1,\end{cases}\quad\text{and}\quad C=-7\overline{d_{1\{A\}}}. \tag{5.8}$$

We denote $\mathrm{Arg}(Q_j \to P_j;\ell)$ in the following way: suppose $P_j(d_1) = Re^{i\Theta}$ and $Q_j(B) = R_B e^{i\Theta_B}$, then

$$\mathrm{Arg}(Q_j \to P_j;\ell) = \alpha \quad \text{if and only if} \quad \Theta - \Theta_B = \alpha\cdot 2\pi + 2k\pi \text{ for } k\in\mathbb{Z}.$$

We also denote $\mathrm{Arg}(Q \to P;\ell) = \sum_{j=1}^{3}\mathrm{Arg}(Q_j \to P_j;\ell)$. Note that if $\mathrm{Arg}(Q_j \to P_j;\ell) = \alpha$, then $\mathrm{Arg}(Q_j \to P_j;\ell) = \alpha + k$ for all $k\in\mathbb{Z}$.

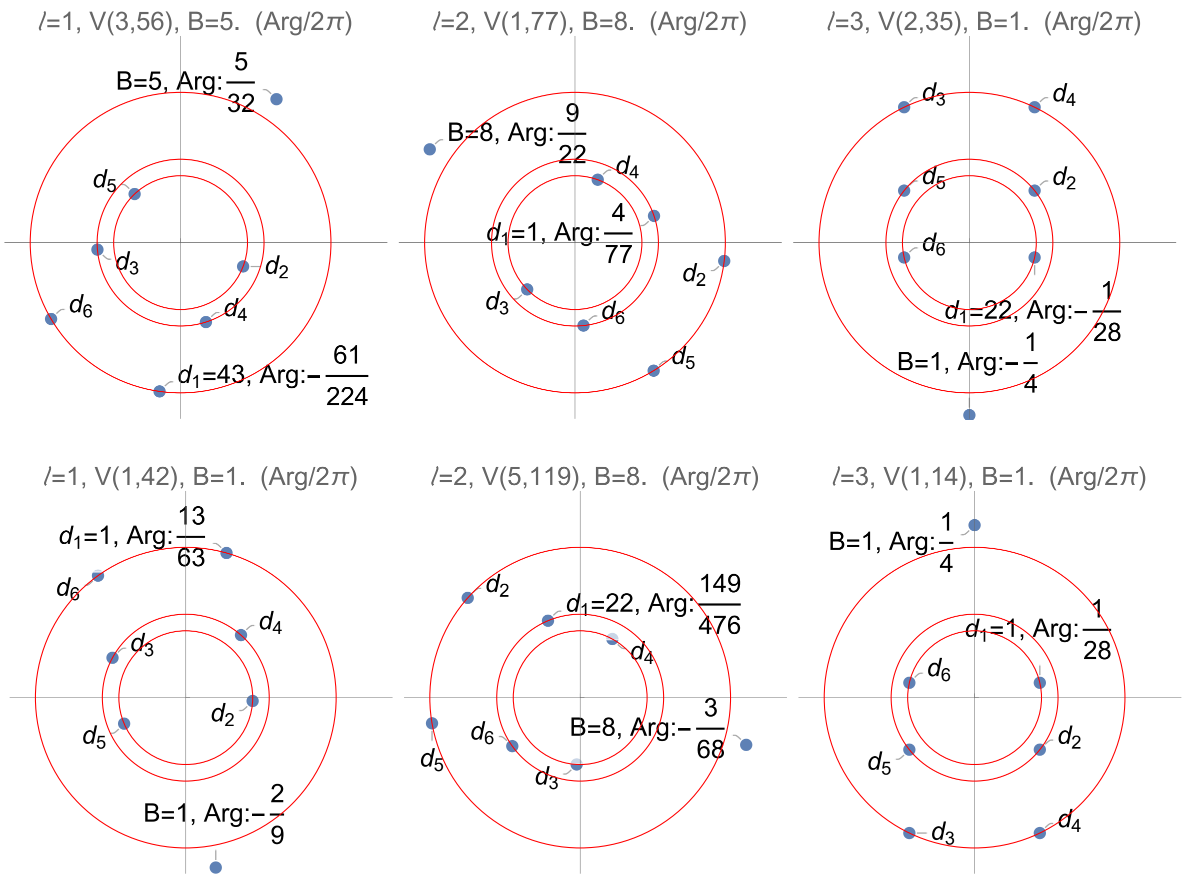
With the notation above, we claim that the argument differences satisfy the following proposition.

Proposition 5.1. *For $c = 7c' = 7A$, any $r \pmod{c'}^*$, $d_1 \in V(r,c)$ and B chosen by (5.8), we have*

$$A\ell = 7T+1: \quad \mathrm{Arg}(Q\to P;\ell) = -\frac{3}{7}, -\frac{5}{14}, \frac{3}{14} \quad \textit{for } \ell = 1,2,3; \tag{5.9}$$

$$A\ell = 7T-1: \quad \mathrm{Arg}(Q\to P;\ell) = \frac{3}{7}, \frac{5}{14}, -\frac{3}{14} \quad \textit{for } \ell = 1,2,3. \tag{5.10}$$

To visualize the argument differences, here are a few examples:



The red circles in the figures are centered at the origin with radii $\csc(\frac{\pi}{7})$, $\csc(\frac{2\pi}{7})$, and $\csc(\frac{3\pi}{7})$, respectively, from the outside to the inside. The point labeled by B represents $\frac{Q(B)}{2}$.

For the styles of the six points $P(d_j)$ for $d_j \in V(r,c)$, we have the following condition. This has already been proved by the tables in the former section, corresponding to the rows marked with "$c'\ell \equiv \pm 1 \pmod 7$?" whose entries are $+$ or $-$.

Condition 5.2. *When $c'\ell \equiv \pm 1 \pmod 7$, we have the following six styles for these six points $P(d)$ for $d \in V(r,c)$.*

- $\ell = 1$. *When $c'\ell \equiv 1 \pmod 7$, the arguments going $d_1 \to d_2 \to d_3 \to d_4 \to d_5 \to d_6 \to d_1$ are $\frac{3}{14}$, $-\frac{3}{7}$, $\frac{2}{7}$, $-\frac{3}{7}$, $\frac{3}{14}$, $\frac{1}{7}$, respectively. When $c'\ell \equiv -1 \pmod 7$, the direction is reversed, as shown in the second line.*

	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$c' \equiv 1 \pmod 7$	$\frac{3}{14}$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{3}{14}$	$\frac{1}{7}$
$c' \equiv 6 \pmod 7$	$-\frac{3}{14}$	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{3}{14}$	$-\frac{1}{7}$

- $\ell = 2$. *The first line is for $c'\ell \equiv 1 \pmod 7$ and the second line is for $c'\ell \equiv -1 \pmod 7$.*

	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$c' \equiv 4 \pmod 7$	$-\frac{1}{14}$	$-\frac{5}{14}$	$-\frac{3}{7}$	$-\frac{5}{14}$	$-\frac{1}{14}$	$\frac{2}{7}$
$c' \equiv 3 \pmod 7$	$\frac{1}{14}$	$\frac{5}{14}$	$\frac{3}{7}$	$\frac{5}{14}$	$\frac{1}{14}$	$-\frac{2}{7}$

- $\ell = 3$. *The first line is for $c'\ell \equiv 1 \pmod 7$ and the second line is for $c'\ell \equiv -1 \pmod 7$.*

	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$c' \equiv 5 \pmod 7$	$\frac{1}{7}$	$\frac{3}{14}$	$-\frac{1}{7}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{3}{7}$
$c' \equiv 2 \pmod 7$	$-\frac{1}{7}$	$-\frac{3}{14}$	$\frac{1}{7}$	$-\frac{3}{14}$	$-\frac{1}{7}$	$-\frac{3}{7}$

If the six points $P(d)$ for $d \in V(r,c)$ satisfy Condition 5.2, and $\mathrm{Arg}(Q \to P; \ell)$ satisfies (5.9) and (5.10) in the corresponding cases, then we have

$$s_{r,c} := \sum_{d \in V(r,c)} P(d) + Q(B) = 0. \tag{5.11}$$

Note that B is chosen from $d_1 \in V(r,c)$ and A, hence from r and c. One way is by using

$$\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})} + \frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})} - \frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})} = \sqrt{7}, \quad \text{where } \frac{1}{\sin(\frac{j\pi}{7})} \text{ for } j = 1,2,3 \text{ are the radii.}$$

Proof of (7-5,2) of Theorem 1.3. This is implied by (5.1), which is proved by (5.11), (5.2), (5.3), (5.4), $7B \equiv r \pmod A$, and

$$\begin{aligned} &e(\tfrac{1}{8}) S^{(\ell)}_{\infty\infty}(0, 7n+5, c, \mu_7) + 2i\sqrt{7} S^{(\ell)}_{0\infty}(0, 7n+5, A, \mu_7; 0) \\ &= (-1)^{A\ell - [A\ell]} \sum_{r \pmod A)^*} s_{r,c}\, e\left(\frac{nr}{A}\right) = 0. \end{aligned}$$

□

Subsections §5.1-§5.4 are devoted to prove (5.9), i.e. the cases $A\ell = c'\ell \equiv 1 \pmod 7$. We will not repeat the proof for (5.10) but just list a few key intermediate steps at the end.

5.1. **Case** $c'\ell \equiv 1 \pmod 7$, $2 \nmid A$, **and** $3 \nmid A$**.** Recall $d_1 \equiv 1 \pmod 7$ and $d_1 \equiv r \pmod{c'}$. Recall that we define $1 \le \beta \le 6$ as $\beta c' \equiv 1 \pmod 7$ and here $\beta = \ell$. Note that $d_1 - \beta A \equiv 7B \pmod{7A}$:

$$7B = d_1(1 - A\ell) \equiv d_1 + (7 - d_1)\ell A \pmod{7A}, \text{ so } 7B \equiv \begin{cases} 0 \pmod 7, \\ r \pmod A). \end{cases}$$

On the other hand, $d_1 - \beta c' \equiv r \pmod A$ and $d_1 - \beta c' \equiv 0 \pmod 7$. The argument difference between P_3 and Q_3 is easy to compute:

$$7\,\mathrm{Arg}(Q_3 \to P_3; \ell) \equiv 5 d_1 \ell \equiv 5\ell \pmod 7 \tag{5.12}$$

which does not depend on n.

Recall $\overline{d_{1\{7A\}}} \equiv a_1 \pmod{7A}$ and $\overline{B_{\{A\}}} \equiv 7\overline{d_{1\{A\}}} \pmod{A}$. We have

$$\begin{aligned}-84A(s(d_1,7A)-s(B,A)) &\equiv -d_1-a_1+d_1(1-\beta A)+49\overline{d_{1\{A\}}}\\ &\equiv -d_1\beta A - a_1 + 49\overline{d_{1\{A\}}} \pmod{7A}.\end{aligned}$$

Hence

$$-84A(s(d_1,7A)-s(B,A)) \equiv \begin{cases} -2 \pmod{7} \\ 48\overline{d_{1\{A\}}} \pmod{A} \end{cases} \tag{5.13}$$

We also have

$$\begin{aligned}-84A(s(d_1,7A)-s(B,A)) &\equiv -7A-1+2(\tfrac{d_1}{7A})+7(A+1)-14(\tfrac{B}{A})\\ &\equiv 6+2(\tfrac{d_1}{A})+2(\tfrac{d_1}{A})(\tfrac{7}{A}) \pmod{8},\end{aligned}$$

where the last step is because $(A,7)=1$, $(\frac{d_1}{7})=(\frac{1}{7})=1$ and $7B\equiv d_1 \pmod{A}$. By A is odd and $A\ell\equiv 1 \pmod{T}$, we have $(\frac{7}{A})=(\frac{\ell}{7})(-1)^{\frac{A-1}{2}}$. Combining $6|12cs(d_1,c)$ and $6|12As(B,A)$ we conclude

$$-84A(s(d_1,7A)-s(B,A)) \equiv \begin{cases} 18 \pmod{24}, & \text{if } \quad A\equiv 1 \pmod{4} \text{ which requires:}\\ & \qquad\quad \ell=1,\ 4|T;\\ & \qquad\quad \ell=2,\ T\equiv 7 \pmod{8};\\ & \text{or if } A\equiv 3 \pmod{4} \text{ which requires:}\\ & \qquad\quad \ell=3,\ T\equiv 8 \pmod{12};\\ \\ 6 \pmod{24}, & \text{if } \quad A\equiv 3 \pmod{4} \text{ which requires:}\\ & \qquad\quad \ell=1,\ 2\|T;\\ & \qquad\quad \ell=2,\ T\equiv 3 \pmod{8};\\ & \text{or if } A\equiv 1 \pmod{4} \text{ which requires:}\\ & \qquad\quad \ell=3,\ T\equiv 2 \pmod{12}.\end{cases} \tag{5.14}$$

Next we check the part of Q_2 other than $e^{-\pi i s(B,A)}$. Since A is odd and T is even, we have

$$\begin{aligned}\left(\tfrac{3}{2}T^2+\tfrac{1}{2}T\right)C &\equiv \tfrac{T}{2}(3T+1)(-7\overline{d_{1\{A\}}})\\ &\equiv \tfrac{T}{2}(3-3A\ell-7)\overline{d_{1\{A\}}}\\ &\equiv -2T\overline{d_{1\{A\}}} \pmod{A}.\end{aligned}$$

Then the part of Q_2 other than $e^{-\pi i s(d,c)}$ is

$$e\left(\frac{24\cdot 2\overline{d_{1\{A\}}}(-7T)}{24\cdot 7A}\right) = e\left(\frac{48\overline{d_{1\{A\}}}(1-A\ell)}{168A}\right), \text{ with numerator } \equiv \begin{cases} 0 \pmod{7},\\ 48\overline{d_{1\{A\}}} \pmod{A},\\ 0 \pmod{24}.\end{cases} \tag{5.15}$$

We conclude that

$$24\cdot 7A\,\mathrm{Arg}(Q_2\to P_2;\ell) \equiv R_2 \pmod{168A} \tag{5.16}$$

where R_2 is determined by (5.13), (5.14) and (5.15): $R_2\equiv 0 \pmod{A}$, $R_2\equiv -2 \pmod{7}$, and $R_2\equiv 18,6 \pmod{24}$ depending on the cases in (5.14). Therefore, by $A\ell\equiv 1 \pmod{7}$ and $A \pmod{4}$ in (5.14) we conclude

$$\mathrm{Arg}(Q_2\to P_2;\ell) = \frac{23,\ 11,\ 13}{28} \quad \text{for } \ell=1,2,3. \tag{5.17}$$

Then we compute $\mathrm{Arg}(Q_1 \to P_1;\ell)$. When $\ell = 1$, since A is odd, $A \equiv 1 \pmod{14}$. Note that both $a_1 \equiv 1, 8 \pmod{14}$ give the same result due to the sign of $\sin(\frac{\pi a}{7})$. It is direct to get

$$\mathrm{Arg}(Q_1 \to P_1; 1) = \frac{1}{2} - \frac{3}{14} - \frac{1}{4} = \frac{1}{28}. \tag{5.18}$$

When $\ell = 2$, we get $A \equiv 4 \pmod 7$ and

$$\mathrm{Arg}(Q_1 \to P_1; 2) = \frac{1}{2} - \frac{3}{7} - \frac{1}{4} = -\frac{5}{28}. \tag{5.19}$$

When $\ell = 3$, we have $A \equiv 5 \pmod{14}$ and

$$\mathrm{Arg}(Q_1 \to P_1; 3) = \frac{1}{2} - \frac{9}{14} - \frac{1}{4} = -\frac{11}{28}. \tag{5.20}$$

Combining (5.18), (5.19), (5.20), (5.17), and (5.12) proves (5.9).

5.2. **Case $c'\ell \equiv 1 \pmod 7$, $2 \nmid A$, and $3|A$.** In this case (5.12) still holds. For $\mathrm{Arg}(Q_2 \to P_2;\ell)$, by (2.3) we have

$$-84A(s(d_1,7A) - s(B,A)) \equiv -d_1A\ell - \overline{d_{1\{21A\}}} + 7\overline{(-d_1T)_{\{3A\}}} \pmod{21A}.$$

We have

$$\begin{aligned}
-84A(s(d_1,7A) - s(B,A)) &\equiv -d_1A\ell - \overline{d_{1\{3A\}}} + 49\overline{(d_1 - d_1A\ell)_{\{3A\}}} \\
&\equiv -d_1A\ell + (48d_1 + d_1A\ell)\overline{d_{1\{3A\}}}\,\overline{(d_1 - d_1A\ell)_{\{3A\}}} \\
&\equiv d_1A\ell\left(\overline{d_{1\{3A\}}}\,\overline{(d_1 - d_1A\ell)_{\{3A\}}} - 1\right) + 48\overline{d_{1\{A\}}} \\
&\equiv 48\overline{d_{1\{A\}}} \pmod{3A}
\end{aligned} \tag{5.21}$$

where in the second congruence we use

$$\overline{(x+y)_m} - 49\overline{x_{\{m\}}} \equiv \overline{x_{\{m\}}}\,\overline{(x+y)_{\{m\}}}(-48x - 49y) \pmod m$$

for $(x+y,m) = (x,m) = 1$ and in the last two congruences we use

$$m_1\overline{x_{\{m_1m_2\}}} \equiv m_1\overline{x_{\{m_2\}}} \pmod{m_1m_2} \tag{5.22}$$

for $(x, m_1m_2) = 1$. We still have

$$-\,84A(s(d_1,7A) - s(B,A)) \equiv -2 \pmod 7. \tag{5.23}$$

Moreover, (5.14) and (5.15) still hold except the second congruence in (5.15) should be changed to $48\overline{d_{1\{A\}}} \pmod{3A}$.

We conclude

$$24 \cdot 7A\,\mathrm{Arg}(Q_2 \to P_2;\ell) \equiv R_2 \pmod{168A} \tag{5.24}$$

where R_2 is determined by (5.21), (5.23), (5.14) and (5.15): $R_2 \equiv 0 \pmod{3A}$, $R_2 \equiv -2 \pmod 7$, and $R_2 \equiv 18, 6 \pmod{24}$ depending on the cases in (5.14). Therefore, by $A\ell \equiv 1 \pmod 7$ and (5.14) we conclude

$$\mathrm{Arg}(Q_2 \to P_2;\ell) = \frac{23,\ 11,\ 13}{28} \quad \text{for } \ell = 1, 2, 3. \tag{5.25}$$

The condition $3|A$ does not affect $\mathrm{Arg}(Q_1 \to P_1;\ell)$ and $\mathrm{Arg}(Q_3 \to P_3;\ell)$. Combining (5.25) with (5.18), (5.19), (5.20), and (5.12), we have proved (5.9) in this case.

5.3. **Case** $c'\ell \equiv 1 \pmod 7$, $2|A$, **and** $3 \nmid A$. Recall (5.12). For $\mathrm{Arg}(Q_2 \to P_2;\ell)$ we have (5.13) and need to use (2.5). Let $\lambda \geq 1$ be defined as $2^\lambda \| A$. Recall $B = -d_1T$ and $7T+1 = A\ell$. We have

$$\begin{aligned}
&-84A(s(d_1,7A)-s(B,A))\\
&\quad\equiv -d_1 - \overline{d_1}_{\{8\times 2^\lambda\}}(49A^2+21A+1) - 14\overline{d_1}_{\{8\times 2^\lambda\}}A\left(\tfrac{7A}{d_1}\right)\\
&\qquad + d_1(1-A\ell) + 49\overline{(d_1-d_1A\ell)}_{\{8\times 2^\lambda\}}(A^2+3A+1) + 14\overline{B}_{\{8\times 2^\lambda\}}A\left(\tfrac{A}{B}\right)\\
&\quad\equiv -d_1A\ell + 49A^2\cdot d_1A\ell\overline{(d_1-d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}}\\
&\qquad + 21A(6d_1+d_1A\ell)\overline{(d_1-d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}}\\
&\qquad + (48d_1+d_1A\ell)\overline{(d_1-d_1A\ell)}_{\{8\times 2^\lambda\}}\overline{d_1}_{\{8\times 2^\lambda\}}\\
&\qquad + 14A\left(\overline{B}_{\{8\times 2^\lambda\}}\left(\tfrac{A}{B}\right) - \overline{d_1}_{\{8\times 2^\lambda\}}\left(\tfrac{7A}{d_1}\right)\right) \pmod{8\times 2^\lambda}.
\end{aligned}$$

Since $2^\lambda \| A$ with $\lambda \geq 1$, we apply (5.22) and $x^2 \equiv 1 \pmod 8$ for odd x to get

$$\begin{aligned}
-84A(s(d_1,7A)-s(B,A)) \equiv{}& 6d_1A + d_1A^2\ell(1+\ell) + 48\overline{d_1}_{\{A\}}\\
&+ 6A\left(B\left(\tfrac{A}{B}\right) - d_1\left(\tfrac{7A}{d_1}\right)\right) \pmod{8\times 2^\lambda}.
\end{aligned}$$

By (5.23), To determine $B(\frac{A}{B}) - d_1(\frac{7A}{d_1}) \pmod 4$, we use the quadratic reciprocity (2.6). By $B<0$ odd and $A>0$, we compute

$$\begin{aligned}
B\left(\tfrac{A}{B}\right) - d_1\left(\tfrac{7A}{d_1}\right) &\equiv -d_1T\left(\tfrac{B}{A}\right)(-1)^{\frac{\frac{A}{2^\lambda}-1}{2}\cdot\frac{B-1}{2}} - d_1\left(\tfrac{d_1}{7A}\right)(-1)^{\frac{7\cdot\frac{A}{2^\lambda}-1}{2}\cdot\frac{d_1-1}{2}}\\
&\equiv -d_1T\left(\tfrac{d_1-d_1A\ell}{A}\right)\left(\tfrac{7}{A}\right)(-1)^{\frac{\frac{A}{2^\lambda}-1}{2}\cdot\frac{B-1}{2}} - d_1\left(\tfrac{d_1}{A}\right)(-1)^{\frac{7\cdot\frac{A}{2^\lambda}-1}{2}\cdot\frac{d_1-1}{2}} \pmod 4
\end{aligned} \tag{5.26}$$

Here are the cases:

(1) If $4|A$, then we have $T \equiv 1 \pmod 4$, $B \equiv -d_1 \pmod 4$. Moreover, $(\frac{d_1-d_1A\ell}{A}) = (\frac{d_1}{A})$ always (note that A is even and we have to consider $(\frac{d_1}{2})$). Now (5.26) simplifies to $(\frac{\ell}{7})d_1 + 1 \pmod 4$. In this case $d_1A^2\ell(1+\ell) \equiv 0 \pmod{8\times 2^\lambda}$ and we conclude

$$-84A(s(d_1,7A)-s(B,A)) \equiv \begin{cases} 2A + 48\overline{d_1}_{\{A\}} \pmod{8\times 2^\lambda}, & \ell = 1,2;\\ 6A + 48\overline{d_1}_{\{A\}} \pmod{8\times 2^\lambda}, & \ell = 3. \end{cases} \tag{5.27}$$

(2) If $2\|A$ and $\ell = 1$, then $T \equiv 3 \pmod 4$, $B \equiv d_1 \pmod 4$ and the above (5.26) simplifies to $d_1 - 1 \pmod 4$. Then as $A(12d_1 - 6 + 2d_1A) \equiv 2A \pmod{8\times 2^\lambda}$, we conclude the same as the first line of (5.27).

(3) If $2\|A$ and $\ell = 2$, then $T \equiv 1 \pmod 4$, $B \equiv -d_1 \pmod 4$, and $(\frac{d_1-d_1A\ell}{A}) = -(\frac{d_1}{A})$. Now (5.26) gives $d_1 - 1 \pmod 4$ and we again get the first line of (5.27).

(4) If $2\|A$ and $\ell = 3$, then $T \equiv 3 \pmod 4$, $B \equiv d_1 \pmod 4$, and $(\frac{A}{7}) = (\frac{3A}{7})(\frac{3}{7}) = -1$. Here (5.26) results in $d_1 - 1 \pmod 4$ again. Note that $d_1A^2\ell(1+\ell) \equiv 0 \pmod{8\times 2^\lambda}$ and we get the second line of (5.27).

Next we check the part of Q_2 other than $e^{-\pi i s(d,c)}$. In this case A is even, so $3T+1$ is even and we have

$$\left(\tfrac{3}{2}T^2 + \tfrac{1}{2}T\right)C \equiv \tfrac{3T+1}{2}\cdot T(-7\overline{d_1}_{\{A\}}) \equiv \frac{3T+1}{2}\overline{d_1}_{\{A\}} \pmod A.$$

When written with denominator $24 \cdot 7A$, we have

$$e\left(\frac{(\frac{3}{2}T^2+\frac{1}{2}T)C}{A}\right) = e\left(\frac{36A\ell\overline{d_{1\{A\}}}+48\overline{d_{1\{A\}}}}{24\cdot 7A}\right)$$

whose numerator is

$$36A\ell\overline{d_{1\{A\}}}+48\overline{d_{1\{A\}}} \equiv \begin{cases} 0 \pmod 7, & \\ 48\overline{d_{1\{A\}}} \pmod{3A}, & \\ 4A+48\overline{d_{1\{A\}}} \pmod{8\times 2^\lambda}, & \ell=1,3, \\ 48\overline{d_{1\{A\}}} \pmod{8\times 2^\lambda}, & \ell=2. \end{cases} \tag{5.28}$$

Combining the above computation with (5.13), (5.27) and (2.2), we get

$$\operatorname{Arg}(Q_2\to P_2;\ell) = \frac{9,11,27}{28} \quad \text{for } \ell=1,2,3. \tag{5.29}$$

Then we compute $\operatorname{Arg}(Q_1\to P_1;\ell)$. When $\ell=1$, since A is even, $\frac{A}{2}\equiv 4 \pmod 7$. Note that $a_1\equiv 1 \pmod{14}$ because a_1 is odd. It is direct to get (remember $Q_1=i$)

$$\operatorname{Arg}(Q_1\to P_1;1) = \frac{1}{2}-\frac{5}{7}-\frac{1}{4} = -\frac{13}{28}. \tag{5.30}$$

When $\ell=2$, we get $\frac{A}{2}\equiv 2 \pmod 7$ and

$$\operatorname{Arg}(Q_1\to P_1;2) = \frac{1}{2}-\frac{3}{7}-\frac{1}{4} = -\frac{5}{28}. \tag{5.31}$$

When $\ell=3$, we have $\frac{A}{2}\equiv 6 \pmod{14}$ and

$$\operatorname{Arg}(Q_1\to P_1;3) = \frac{1}{2}-\frac{1}{7}-\frac{1}{4} = \frac{3}{28}. \tag{5.32}$$

Combining (5.30), (5.31), (5.32), (5.29), and (5.12), we get

$$\operatorname{Arg}(Q\to P;\ell) = -\frac{3}{7},-\frac{5}{14},\frac{3}{14} \quad \text{for } \ell=1,2,3. \tag{5.33}$$

This proves (5.9).

5.4. **Case** $c'\ell\equiv 1 \pmod 7$, $2|A$, **and** $3|A$**.** Comparing to the former case, the only difference in getting $\operatorname{Arg}(Q_2\to P_2;\ell)$ in (5.29) is that we should using (5.21) instead of (5.13). The result (5.29) still holds in this case. The condition $3|A$ or $3\nmid A$ does not affect the computation for $\operatorname{Arg}(Q_1\to P_1;\ell)$ and $\operatorname{Arg}(Q_3\to P_3;\ell)$, hence we still have (5.9):

$$\operatorname{Arg}(Q\to P;\ell) = -\frac{3}{7},-\frac{5}{14},\frac{3}{14} \quad \text{for } \ell=1,2,3. \tag{5.34}$$

Now we have finished the discussion in all the cases for A when $A\ell\equiv 1 \pmod 7$ and proved (5.9) in Proposition 5.1. For the other case $A\ell\equiv -1 \pmod 7$, we will not repeat the same process but just list the key argument differences below. For every $r \pmod{c'}^*$, we compare $P(d_1)$ (5.5) given $d_1\in V(r,c)$ and $Q(B)$ (5.7) given

$$T := \frac{A\ell+1}{7} > 0, \quad B=d_1T \quad \text{and} \quad C=-7\overline{d_{1\{A\}}}.$$

Now $7B=d_1+d_1A\ell$. We shall get Table 5.1.

We have finished the proof of (7-5,2) of Theorem 1.3.

Case $2\nmid A$:	$\ell=1$	$\ell=2$	$\ell=3$
$\mathrm{Arg}(Q_1\to P_1;\ell)$	$-\frac{1}{28}$	$\frac{5}{28}$	$\frac{11}{28}$
$\mathrm{Arg}(Q_2\to P_2;\ell)$	$\frac{5}{28}$	$-\frac{11}{28}$	$-\frac{13}{28}$
$\mathrm{Arg}(Q_3\to P_3;\ell)$	$\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$
$\mathrm{Arg}(Q\to P;\ell)$	$\frac{3}{7}$	$\frac{5}{14}$	$-\frac{3}{14}$
Case $2\mid A$:	$\ell=1$	$\ell=2$	$\ell=3$
$\mathrm{Arg}(Q_1\to P_1;\ell)$	$\frac{13}{28}$	$\frac{5}{28}$	$-\frac{3}{28}$
$\mathrm{Arg}(Q_2\to P_2;\ell)$	$-\frac{9}{28}$	$-\frac{11}{28}$	$\frac{1}{28}$
$\mathrm{Arg}(Q_3\to P_3;\ell)$	$\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$
$\mathrm{Arg}(Q\to P;\ell)$	$\frac{3}{7}$	$\frac{5}{14}$	$-\frac{3}{14}$

Table 5.1. Table for the case $A\ell\equiv -1 \pmod 7$

6. Part (ii) of Theorem 1.3

For prime $p=5,7$ and integers a,b, recall the notation

$$C_p^{a,b}=\cos(\tfrac{a\pi}{p})-\cos(\tfrac{b\pi}{p}).$$

6.1. **(5-1) and (5-2) of Theorem 1.3.** We still denote $c'=c/5$ and first deal with the case $25|c$. For $(r,c)=1$, recall (3.47) with $V(r,c)$, d and d_* in that subsection. By (3.47), we have $\mathrm{Arg}(d\to d_*;\ell)=-\frac{1}{5}$ for the $5n+4$ case. Since we have the $5n+1$ case here, we obtain

$$\mathrm{Arg}(d\to d_*;\ell)=-\tfrac{1}{5}-\tfrac{3c'}{c}=-\tfrac{4}{5}.$$

Hence we get $S^{(\ell)}_{\infty\infty}(0,5n+1,c,\mu_5)=0$ for $\ell\in\{1,2\}$, $25|c$, and every $n\ge 0$. This proves (5-1) when $25|c$.

Similarly, in the $5n+2$ case, we obtain

$$\mathrm{Arg}(d\to d_*;\ell)=-\tfrac{1}{5}-\tfrac{2c'}{c}=-\tfrac{3}{5}.$$

We still get $S^{(\ell)}_{\infty\infty}(0,5n+2,c,\mu_5)=0$ for $\ell\in\{1,2\}$, $25|c$, and every $n\ge 0$, which proves (5-2) when $25|c$.

Now we focus on the case $5\|c$. For $(r,c)=1$, recall the notation of $V(r,c)$, d_j and a_j in (3.3). By (3.5) and Condition 3.2, we find that the argument differences of $P(d)$ for $d\in V(r,c)$ only depends on $c' \pmod 5$. Moreover, in (3.1), if we change $5n+4$ to $5n+1$ or to $5n+2$, then only the argument of $P_3(d)$ is affected. Recall that we define $\beta\in\{1,2,3,4\}$ by $\beta c'\equiv 1 \pmod 5$.

6.1.1. *The $5n+1$ case.* As Proposition 3.1, we denote

$$s^{(\ell)}_{r,c}=\sum_{d\in V(r,c)}P_1(d)P_2(d)P_3(d):=\sum_{d\in V(r,c)}\frac{e\left(-\frac{3c'a\ell^2}{10}\right)}{\sin(\frac{\pi a\ell}{5})}e^{-\pi i s(d,c)}e\left(\frac{d}{c}\right). \tag{6.1}$$

Here $P_3(d)=e(\frac{d}{c})$ instead of $e(\frac{4d}{c})$ in the $5n+4$ case Proposition 3.1. To prove (5-1) of Theorem 1.3, it suffices to show

$$C_5^{2,4}\sin(\tfrac{\pi}{5})s^{(1)}_{r,c}+C_5^{4,2}\sin(\tfrac{2\pi}{5})s^{(2)}_{r,c}=0. \tag{6.2}$$

When we compute $\mathrm{Arg}_3(d_j \to d_{j+1}; \ell)$ for $j = 1, 2, 3$, previously it was $e(\frac{4\beta}{5})$ and now it should be $e(\frac{\beta}{5})$. Therefore, we need to subtract $\frac{3\beta}{5}$ from the argument differences in Condition 3.2 in each case.

Since we need both $\ell = 1$ and $\ell = 2$ appears at the same time, we write our new condition in the following way. It is important to note that the way we compute $\mathrm{Arg}(d_4 \to d_1; \ell)$ is by

$$\sum_{j=1}^{3} \mathrm{Arg}(d_j \to d_{j+1}; \ell) + \mathrm{Arg}(d_4 \to d_1; \ell) = 0$$

but not by subtracting $\frac{3\beta}{5}$.

Condition 6.1. *For the $5n+1$ case, we have the following styles of argument differences:*

- $c' \equiv 1 \pmod 5$, $\beta = 1$;

$c' \equiv 1 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{3}{10}$		$\frac{1}{2}$		$-\frac{3}{10}$		$\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{2}{5}$		$-\frac{1}{10}$		$\frac{2}{5}$		$\frac{3}{10}$	

- $c' \equiv 2 \pmod 5$, $\beta = 3$;

$c' \equiv 2 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{3}{10}$		$-\frac{3}{10}$		$-\frac{3}{10}$		$-\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{2}{5}$		$\frac{1}{2}$		$-\frac{2}{5}$		$\frac{3}{10}$	

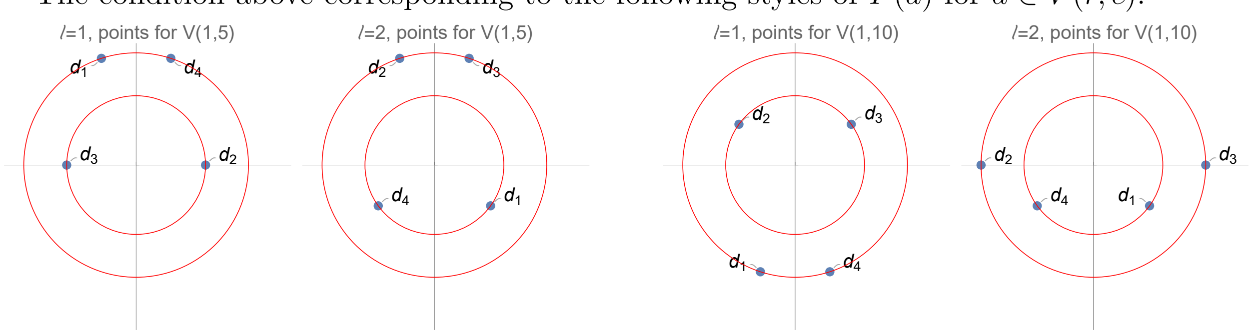
- $c' \equiv 3 \pmod 5$, $\beta = 2$;

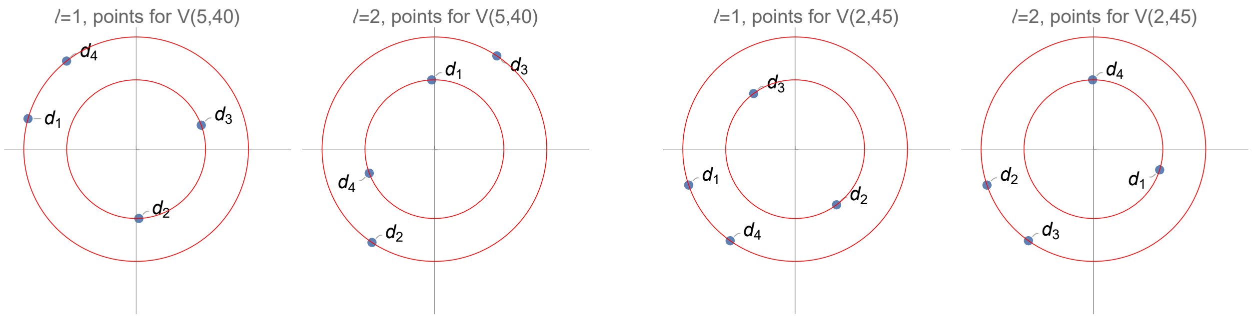
$c' \equiv 3 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{3}{10}$		$\frac{3}{10}$		$\frac{3}{10}$		$\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{2}{5}$		$\frac{1}{2}$		$\frac{2}{5}$		$-\frac{3}{10}$	

- $c' \equiv 4 \pmod 5$, $\beta = 4$.

$c' \equiv 4 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{3}{10}$		$\frac{1}{2}$		$\frac{3}{10}$		$-\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{2}{5}$		$\frac{1}{10}$		$-\frac{2}{5}$		$-\frac{3}{10}$	

The condition above corresponding to the following styles of $P(d)$ for $d \in V(r, c)$:





For every two graphs close to each other in a row which satisfy the argument differences in the corresponding cases in Condition 6.1, it proves (6.1) due to the following equations:

$$C_5^{2,4}\cos(\tfrac{\pi}{10}) + C_5^{4,2}\sin(\tfrac{2\pi}{5})\left(\frac{\cos(\frac{\pi}{10})}{\sin(\frac{\pi}{5})} - \frac{\cos(\frac{3\pi}{10})}{\sin(\frac{2\pi}{5})}\right) = 0, \quad \text{for } c' \equiv 1, 4 \pmod 5;$$

$$C_5^{2,4}\sin(\tfrac{\pi}{5})\left(\frac{\cos(\frac{\pi}{10})}{\sin(\frac{\pi}{5})} - \frac{\cos(\frac{3\pi}{10})}{\sin(\frac{2\pi}{5})}\right) + C_5^{4,2}\cos(\tfrac{3\pi}{10}) = 0, \quad \text{for } c' \equiv 2, 3 \pmod 5.$$

This proves (6.2), hence proves (5-1) of Theorem 1.3.

6.1.2. *The* $5n+2$ *case.* As (6.1), we denote $P_3(d) = e(\frac{2d}{c})$ instead of $e(\frac{4d}{c})$ in the $5n+4$ case Proposition 3.1 and instead of $e(\frac{d}{c})$ in the $5n+1$ case (6.1). To prove (5-2) of Theorem 1.3, it suffices to show

$$C_5^{0,4}\sin(\tfrac{\pi}{5})s_{r,c}^{(1)} + C_5^{0,2}\sin(\tfrac{2\pi}{5})s_{r,c}^{(2)} = 0. \tag{6.3}$$

When we compute $\mathrm{Arg}_3(d_j \to d_{j+1}; \ell)$ for $j = 1, 2, 3$, in (6.1) it was $e(\frac{\beta}{5})$ and now it should be $e(\frac{2\beta}{5})$. Therefore, we need to add $\frac{\beta}{5}$ to the argument differences in Condition 6.1 in each case to get the following condition.

Condition 6.2. *For the* $5n+2$ *case, we have the following styles of argument differences:*

- $c' \equiv 1 \pmod 5$, $\beta = 1$;

$c' \equiv 1 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{1}{10}$		$-\frac{3}{10}$		$-\frac{1}{10}$		$\frac{1}{2}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{2}{5}$		$\frac{1}{10}$		$-\frac{2}{5}$		$-\frac{3}{10}$	

- $c' \equiv 2 \pmod 5$, $\beta = 3$;

$c' \equiv 2 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{3}{10}$		$\frac{3}{10}$		$\frac{3}{10}$		$\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{5}$		$\frac{1}{10}$		$\frac{1}{5}$		$\frac{1}{2}$	

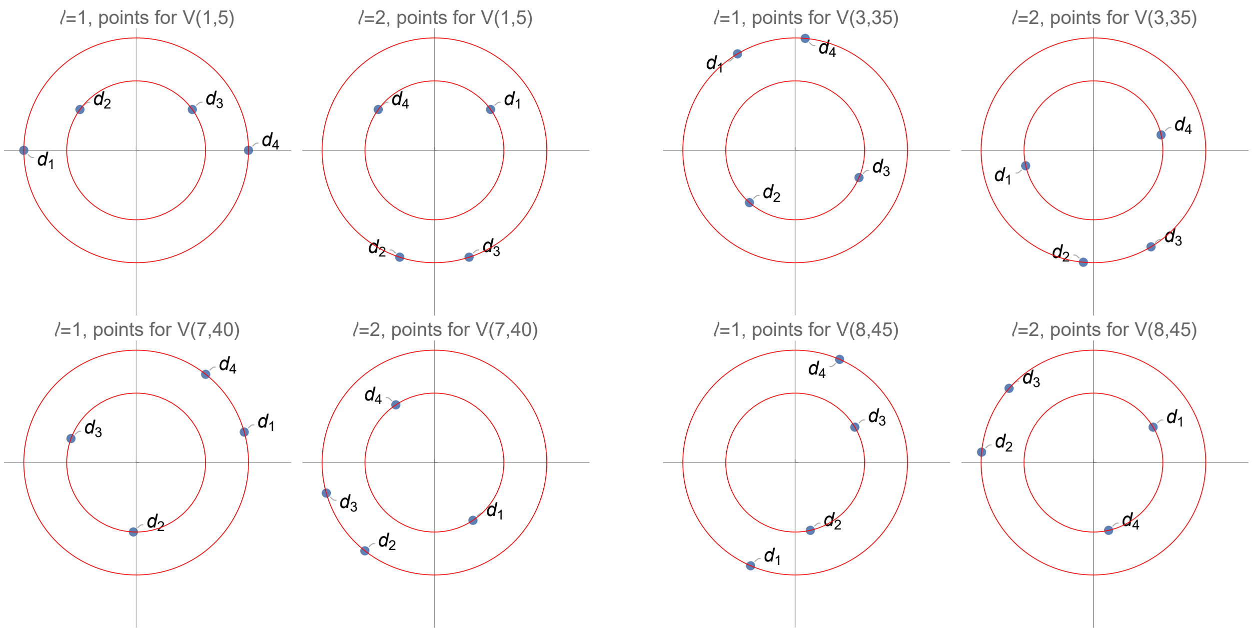
- $c' \equiv 3 \pmod 5$, $\beta = 2$;

$c' \equiv 3 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{3}{10}$		$-\frac{3}{10}$		$-\frac{3}{10}$		$-\frac{1}{10}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{1}{5}$		$-\frac{1}{10}$		$-\frac{1}{5}$		$\frac{1}{2}$	

- $c' \equiv 4 \pmod 5$, $\beta = 4$.

$c' \equiv 4 \pmod 5$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{10}$		$\frac{3}{10}$		$\frac{1}{10}$		$\frac{1}{2}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{2}{5}$		$-\frac{1}{10}$		$\frac{2}{5}$		$\frac{3}{10}$	

The condition above corresponding to the following styles of $P(d)$ for $d \in V(r,c)$.



For every two graphs with same c which satisfy the argument differences in the corresponding cases in Condition 6.2, it proves (6.3) due to the following equations:

$$C_5^{0,4}\sin(\tfrac{\pi}{5})\cdot\frac{\cos(\frac{3\pi}{10})}{\sin(\frac{2\pi}{5})}+C_5^{0,2}\sin(\tfrac{2\pi}{5})\left(\frac{\cos(\frac{3\pi}{10})}{\sin(\frac{2\pi}{5})}-\frac{\cos(\frac{\pi}{10})}{\sin(\frac{\pi}{5})}\right)=0,\quad \text{for } c'\equiv 1,4 \pmod 5;$$

$$C_5^{0,4}\sin(\tfrac{\pi}{5})\left(\frac{\cos(\frac{\pi}{10})}{\sin(\frac{\pi}{5})}-\frac{\cos(\frac{3\pi}{10})}{\sin(\frac{2\pi}{5})}\right)-C_5^{0,2}\sin(\tfrac{2\pi}{5})\cdot\frac{\cos(\frac{\pi}{10})}{\sin(\frac{\pi}{5})}=0,\quad \text{for } c'\equiv 2,3 \pmod 5.$$

This proves (5-2) of Theorem 1.3.

6.2. **Restate the condition for (7-5) of Theorem 1.3.** We still denote $c = 7A = 7c'$. When $49|c$, recall the notation in §4.5 and we have (4.22) for any $d \in V(r,c)$ and $d_* = d + c'$:

$$\mathrm{Arg}(d \to d_*; \ell) = \begin{cases} -\frac{2}{7} & d \equiv 1,6 \pmod 7; \\ \frac{3}{7} & d \equiv 2,5 \pmod 7; \\ -\frac{1}{7} & d \equiv 3,4 \pmod 7. \end{cases} \tag{6.4}$$

When $7\|c'$, denote $A = c' = c/7$ and recall the notation of $V(r,c)$ before (4.1) and d_j and a_j in (4.4). We combine Condition 4.2, Condition 5.2, (5.9) and (5.10) and get the following condition:

Condition 6.3. *For the $7n+5$ case, we have the following conditions on* $\mathrm{Arg}(Q \to P; \ell)$ *when* $A\ell \equiv \pm 1 \pmod 7$ *with tables for* $\mathrm{Arg}_j(d_u \to d_v; \ell)$.

- $c' \equiv 1 \pmod 7$. $A \cdot 1 = 7T + 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{3}{7}$;
- $c' \equiv 2 \pmod 7$. $A \cdot 3 = 7T - 1$, $\mathrm{Arg}(Q \to P; 3) = -\frac{3}{14}$;

$c' \equiv 1 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$\frac{3}{14}$		$-\frac{3}{7}$		$\frac{2}{7}$		$-\frac{3}{7}$		$\frac{3}{14}$		$\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$-\frac{3}{14}$		$-\frac{1}{14}$		$-\frac{2}{7}$		$-\frac{1}{14}$		$-\frac{3}{14}$		$-\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$-\frac{3}{7}$		$\frac{5}{14}$		$\frac{3}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{2}{7}$	

$c' \equiv 2 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$-\frac{5}{14}$		$-\frac{2}{7}$		$-\frac{1}{7}$		$-\frac{2}{7}$		$-\frac{5}{14}$		$\frac{3}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$-\frac{3}{14}$		$-\frac{1}{14}$		$-\frac{2}{7}$		$-\frac{1}{14}$		$-\frac{3}{14}$		$-\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$-\frac{1}{7}$		$-\frac{3}{14}$		$\frac{1}{7}$		$-\frac{3}{14}$		$-\frac{1}{7}$		$-\frac{3}{7}$	

- $c' \equiv 3 \pmod 7$. $A \cdot 2 = 7T - 1$, $\text{Arg}(Q \to P; 2) = \frac{5}{14}$;

$c' \equiv 3 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$\frac{5}{14}$		$\frac{2}{7}$		$\frac{1}{7}$		$\frac{2}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$\frac{1}{14}$		$\frac{5}{14}$		$\frac{3}{7}$		$\frac{5}{14}$		$\frac{1}{14}$		$-\frac{2}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$\frac{3}{7}$		$-\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{5}{14}$		$\frac{3}{7}$		$\frac{2}{7}$	

- $c' \equiv 4 \pmod 7$. $A \cdot 2 = 7T + 1$, $\text{Arg}(Q \to P; 2) = -\frac{5}{14}$;

$c' \equiv 4 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$-\frac{5}{14}$		$-\frac{2}{7}$		$-\frac{1}{7}$		$-\frac{2}{7}$		$-\frac{5}{14}$		$-\frac{3}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$-\frac{1}{14}$		$-\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{5}{14}$		$-\frac{1}{14}$		$\frac{2}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$-\frac{3}{7}$		$\frac{5}{14}$		$\frac{3}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{2}{7}$	

- $c' \equiv 5 \pmod 7$. $A \cdot 3 = 7T + 1$, $\text{Arg}(Q \to P; 3) = \frac{3}{14}$;

$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$\frac{5}{14}$		$\frac{2}{7}$		$\frac{1}{7}$		$\frac{2}{7}$		$\frac{5}{14}$		$-\frac{3}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$\frac{3}{14}$		$\frac{1}{14}$		$\frac{2}{7}$		$\frac{1}{14}$		$\frac{3}{14}$		$\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$\frac{1}{7}$		$\frac{3}{14}$		$-\frac{1}{7}$		$\frac{3}{14}$		$\frac{1}{7}$		$\frac{3}{7}$	

- $c' \equiv 6 \pmod 7$. $A \cdot 1 = 7T - 1$, $\text{Arg}(Q \to P; 1) = \frac{3}{7}$.

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\text{Arg}(d_u \to d_v; 1)$		$-\frac{3}{14}$		$\frac{3}{7}$		$-\frac{2}{7}$		$\frac{3}{7}$		$-\frac{3}{14}$		$-\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 2)$		$\frac{3}{14}$		$\frac{1}{14}$		$\frac{2}{7}$		$\frac{1}{14}$		$\frac{3}{14}$		$\frac{1}{7}$	
$\text{Arg}(d_u \to d_v; 3)$		$\frac{3}{7}$		$-\frac{5}{14}$		$-\frac{3}{7}$		$-\frac{5}{14}$		$\frac{3}{7}$		$\frac{2}{7}$	

Now we start to prove the (7-k) cases for $k \in \{0, 1, 2, 3, 4, 6\}$.

6.3. **(7-0) of Theorem 1.3.** As (4.1), we still denote $V(r,c) = \{d \pmod{c}^* : d \equiv r \pmod{c)'}\}$, $d_j \in V(r,c)$ by $d_j \equiv j \pmod 7$. Recall the Kloosterman sums defined at (2.7), (2.8) and (2.9).

For $A := c' = c/7$, when $A\ell = 7T+1$ for some integer $T \geq 0$, as (5.6) we define

$$Q(B) := 2\sqrt{7} Q_1(B) Q_2(B) Q_3(B) \tag{6.5}$$

with $Q_1(B) := (-1)^{[A\ell]} i$ and

$$Q_2(B) := e\left(\frac{(\frac{3}{2}T^2 + \frac{1}{2}T)C}{A}\right), \quad \text{and} \quad Q_3(B) = e\left(\frac{0 \cdot B}{A}\right) = 1 \tag{6.6}$$

and let $B = -d_1 T$ with $C = -7\overline{d_{1\{A\}}}$. When $A\ell = 7T-1$ for some $T \geq 0$, we still define $Q(B)$ as above while we take

$$Q_2(B) := e\left(\frac{(\frac{3}{2}(T-1)^2 + \frac{5}{2}T + 1)C}{A}\right), \quad \text{and} \quad B = d_1 T. \tag{6.7}$$

instead. Note that when A is fixed, ℓ is also fixed, i.e. there is only one corresponding $Q(B)$ for every fixed c.

We define the sum on $V(r,c)$ as

$$\begin{gathered} s_{r,c}^{(\ell)} := \sin(\tfrac{\pi\ell}{7}) \sum_{d \in V(r,c)} P_1(d) P_2(d) P_3(d) + \sin(\tfrac{\pi\ell}{7}) \mathbf{1}_{\substack{A:=c/7 \\ [A\ell]=1,6}} Q(B), \quad \text{where} \\ P_1(d) := \frac{(-1)^{\ell c} e\left(-\frac{3c'a\ell^2}{14}\right)}{\sin(\frac{\pi a\ell}{7})}, \quad P_2(d) := e\left(-\frac{12cs(d,c)}{24c}\right), \quad P_3(d) := e\left(\frac{0 \cdot d}{c}\right) = 1. \end{gathered} \tag{6.8}$$

Here $\mathbf{1}_{\text{condition}}$ equals 1 if the condition meets and equals 0 otherwise.

To prove (7-0) of Theorem 1.3, it suffices to show

$$C_7^{4,6} s_{r,c}^{(1)} + C_7^{6,2} s_{r,c}^{(2)} + C_7^{2,4} s_{r,c}^{(3)} = 0. \tag{6.9}$$

First we deal with the case $49|c$ and there is no $Q(B)$. We need to subtract $\frac{5}{7}$ from (6.4) and get

$$\operatorname{Arg}(d \to d_*; \ell) = \begin{cases} 0 & d \equiv 1,6 \pmod 7; \\ -\frac{2}{7} & d \equiv 2,5 \pmod 7; \\ \frac{1}{7} & d \equiv 3,4 \pmod 7. \end{cases}$$

When $r \equiv d \equiv 2,3,4,5 \pmod 7$, we get equi-distribution and (6.9) follows. When $r \equiv d \equiv 1,6 \pmod 7$, note that $P_1(d) = (-1)^{(a+1)c\ell}/\sin(\frac{\pi a\ell}{7})$ for $ad \equiv 1 \pmod c$ have the same $\operatorname{sgn} P_1(d)$ for $\ell = 1,2,3$. Hence every summand for $d \in V(r,c)$ in (6.9) have the same argument and we get (6.9) by

$$C_7^{4,6} \frac{\sin(\frac{\pi}{7})}{\sin(\frac{\pi}{7})} + C_7^{6,2} \frac{\sin(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})} + C_7^{2,4} \frac{\sin(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})} = 0.$$

Next we check the condition for $7\|c$. Comparing with Condition 6.3, since we have different $P_3(d)$ and $Q_3(B)$ in this case, we need to subtract $\frac{5\beta}{7}$ in $\operatorname{Arg}(d_j \to d_{j+1}; \ell)$, $1 \leq j \leq 5$ from

Condition 6.3. We also need to add $\mp\frac{5\ell}{7}$ to $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$. It is important to note that we compute $\mathrm{Arg}(d_6 \to d_1; \ell)$ by

$$\sum_{j=1}^{5} \mathrm{Arg}(d_j \to d_{j+1}; \ell) + \mathrm{Arg}(d_6 \to d_1; \ell) = 0$$

instead of adding $\mp\frac{5\ell}{7}$.

Condition 6.4. *For the $7n$ case, we have the following conditions on $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ with tables for $\mathrm{Arg}_j(d_u \to d_v; \ell)$.*

- $c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{1}{7}$;

$c' \equiv 1 \pmod 7$	$d_1 \to d_2$	$\to d_3$	$\to d_4$	$\to d_5$	$\to d_6$	$\to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{1}{2}$	$-\frac{1}{7}$	$-\frac{3}{7}$	$-\frac{1}{7}$	$\frac{1}{2}$	$-\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{1}{14}$	$\frac{3}{14}$	0	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{1}{7}$	$-\frac{5}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{1}{7}$	$\frac{2}{7}$

- $c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\mathrm{Arg}(Q \to P; 3) = -\frac{1}{14}$;

$c' \equiv 2 \pmod 7$	$d_1 \to d_2$	$\to d_3$	$\to d_4$	$\to d_5$	$\to d_6$	$\to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{3}{14}$	$-\frac{1}{7}$	0	$-\frac{1}{7}$	$-\frac{3}{14}$	$-\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{1}{7}$	$\frac{1}{14}$	$-\frac{1}{14}$	$\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	0	$-\frac{1}{14}$	$\frac{2}{7}$	$-\frac{1}{14}$	0	$-\frac{1}{7}$

- $c' \equiv 3 \pmod 7$. $A \cdot 2 = 7T - 1$, $\mathrm{Arg}(Q \to P; 2) = -\frac{3}{14}$;

$c' \equiv 3 \pmod 7$	$d_1 \to d_2$	$\to d_3$	$\to d_4$	$\to d_5$	$\to d_6$	$\to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{3}{14}$	$\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{1}{2}$	$-\frac{3}{14}$	$-\frac{1}{7}$	$-\frac{3}{14}$	$\frac{1}{2}$	$-\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{1}{7}$	$\frac{1}{14}$	0	$\frac{1}{14}$	$-\frac{1}{7}$	$\frac{1}{7}$

- $c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\mathrm{Arg}(Q \to P; 2) = \frac{3}{14}$;

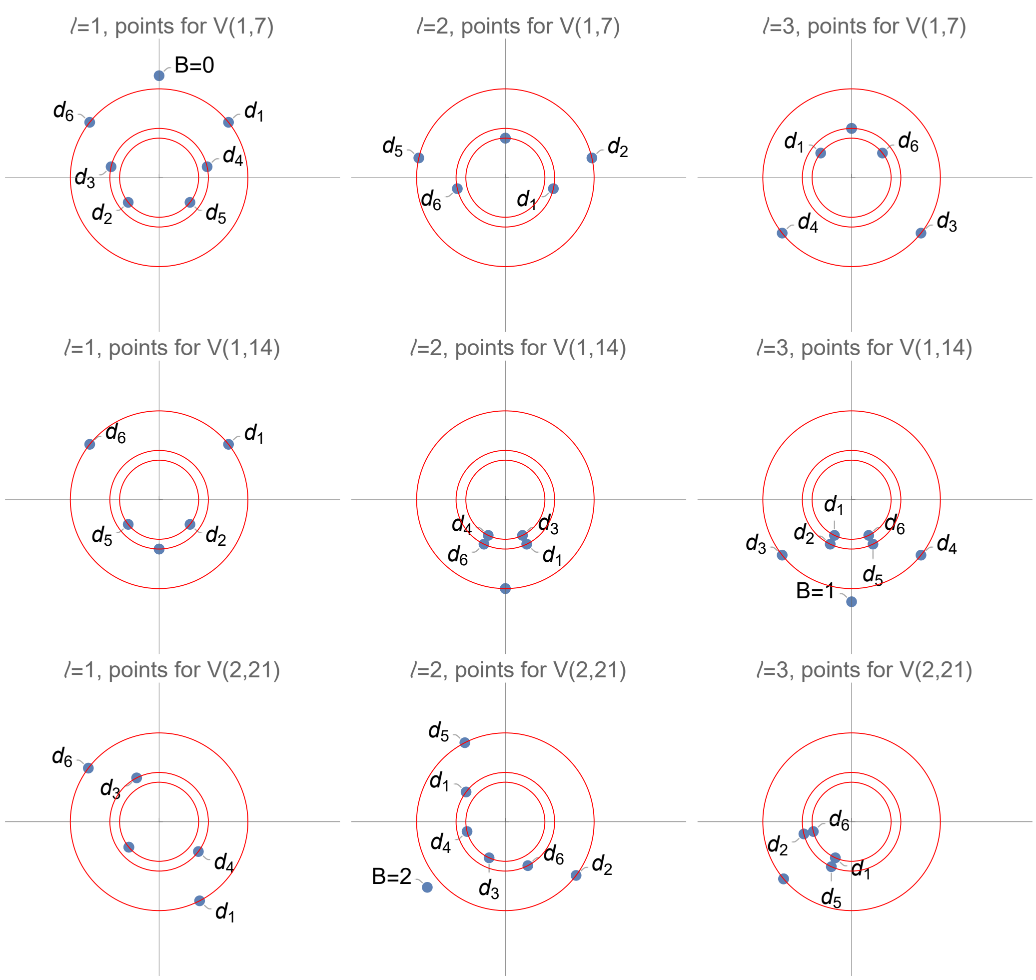
$c' \equiv 4 \pmod 7$	$d_1 \to d_2$	$\to d_3$	$\to d_4$	$\to d_5$	$\to d_6$	$\to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{14}$	$-\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{1}{2}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{1}{2}$	$\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{1}{7}$	$-\frac{1}{14}$	0	$-\frac{1}{14}$	$\frac{1}{7}$	$-\frac{1}{7}$

- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\mathrm{Arg}(Q \to P; 3) = \frac{1}{14}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\mathrm{Arg}(Q \to P; 1) = \frac{1}{7}$.

Note that the condition for $c' \pmod 7$ is the same as the reversed condition for $-c' \pmod 7$. Hence we only need show the corresponding graphs for $c' \equiv 1, 2, 3 \pmod 7$, and also for the other $7n + k$ cases in the remaining subsections. In each of the following graphs, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.

$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{3}{14}$		$\frac{1}{7}$		0		$\frac{1}{7}$		$\frac{3}{14}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{14}$		$-\frac{1}{14}$		$\frac{1}{7}$		$-\frac{1}{14}$		$\frac{1}{14}$		$-\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		0		$\frac{1}{14}$		$-\frac{2}{7}$		$\frac{1}{14}$		0		$\frac{1}{7}$	

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{2}$		$\frac{1}{7}$		$\frac{3}{7}$		$\frac{1}{7}$		$\frac{1}{2}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{1}{14}$		$-\frac{3}{14}$		0		$-\frac{3}{14}$		$-\frac{1}{14}$		$-\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$\frac{1}{7}$		$\frac{5}{14}$		$\frac{2}{7}$		$\frac{5}{14}$		$\frac{1}{7}$		$-\frac{2}{7}$	



Visualizing by the above graphs, (6.9) is proved by the following equations:

$$C_7^{4,6}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{6,2}\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{1}{\sin(\frac{3\pi}{7})}\right)$$
$$+C_7^{2,4}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{4,6}\sin(\tfrac{\pi}{7})\sqrt{7},$$

$$C_7^{4,6}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{6,2}\sin(\tfrac{2\pi}{7})\left(\frac{1}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)$$
$$+C_7^{2,4}\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{2,4}\sin(\tfrac{3\pi}{7})\sqrt{7},$$

$$C_7^{4,6}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{1}{\sin(\frac{3\pi}{7})}\right)+C_7^{6,2}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)$$
$$+C_7^{2,4}\sin(\tfrac{3\pi}{7})\left(\frac{1}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{6,2}\sin(\tfrac{2\pi}{7})\sqrt{7}.$$

This proves (7-0) of Theorem 1.3.

6.4. **(7-1) of Theorem 1.3.** We follow the definition (6.8), (6.6) and (6.7) but we use

$$P_3(d) := e\left(\frac{d}{c}\right) \quad \text{and} \quad Q_3(B) := e\left(\frac{B}{A}\right) \tag{6.10}$$

instead. The following equations suffice to prove (7-1,1) and (7-1,2):

$$\begin{aligned} C_7^{2,4} s_{r,c}^{(1)} + C_7^{4,6} s_{r,c}^{(2)} + C_7^{6,2} s_{r,c}^{(3)} = 0, \\ C_7^{4,6} s_{r,c}^{(1)} + C_7^{6,2} s_{r,c}^{(2)} + C_7^{2,4} s_{r,c}^{(3)} = 0, \end{aligned} \tag{6.11}$$

where $Q(B)$ is determined by $A = c/7$, $A\ell \equiv \pm 1 \pmod 7$ for $\ell \in \{1,2,3\}$ as (6.6) and (6.7).

When $49|c$, there is no $Q(B)$ term. By subtracting $\frac{4}{7}$ from (6.4), $\operatorname{Arg}(d \to d_*; \ell)$ is always a non-zero constant for a fixed $r \pmod{c'}$. Then we get (6.11) by $s_{r,c}^{\ell} = 0$ for $\ell = 1, 2, 3$.

When $7\|c'$, from Condition 6.4 for the $7n$ case, we need to add $\pm\frac{\ell}{7}$ to $\operatorname{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ and add $\frac{\beta}{7}$ to $\operatorname{Arg}(d_j \to d_{j+1}; \ell)$ for $1 \le j \le 5$. We get the following condition.

Condition 6.5. *For the $7n+1$ case, we have the following conditions on $\operatorname{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ with tables for $\operatorname{Arg}_j(d_u \to d_v; \ell)$.*

- $c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\operatorname{Arg}(Q \to P; 1) = 0$;

$c' \equiv 1 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$-\frac{5}{14}$	0	$-\frac{2}{7}$	0	$-\frac{5}{14}$	0
$\operatorname{Arg}(d_u \to d_v; 2)$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{3}{14}$	$-\frac{2}{7}$
$\operatorname{Arg}(d_u \to d_v; 3)$	0	$-\frac{3}{14}$	$-\frac{1}{7}$	$-\frac{3}{14}$	0	$-\frac{3}{7}$

- $c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\operatorname{Arg}(Q \to P; 3) = \frac{1}{2}$;

$c' \equiv 2 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$\frac{5}{14}$	$\frac{3}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{5}{14}$	$-\frac{1}{7}$
$\operatorname{Arg}(d_u \to d_v; 2)$	$\frac{1}{2}$	$-\frac{5}{14}$	$\frac{3}{7}$	$-\frac{5}{14}$	$\frac{1}{2}$	$\frac{2}{7}$
$\operatorname{Arg}(d_u \to d_v; 3)$	$-\frac{3}{7}$	$\frac{1}{2}$	$-\frac{1}{7}$	$\frac{1}{2}$	$-\frac{3}{7}$	0

- $c' \equiv 3 \pmod 7$, $\beta = 5$. $A \cdot 2 = 7T - 1$, $\operatorname{Arg}(Q \to P; 2) = \frac{1}{2}$;

$c' \equiv 3 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{1}{2}$	$-\frac{1}{7}$
$\operatorname{Arg}(d_u \to d_v; 2)$	$\frac{3}{14}$	$\frac{1}{2}$	$-\frac{3}{7}$	$\frac{1}{2}$	$\frac{3}{14}$	0
$\operatorname{Arg}(d_u \to d_v; 3)$	$-\frac{3}{7}$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{3}{14}$	$-\frac{3}{7}$	$-\frac{3}{7}$

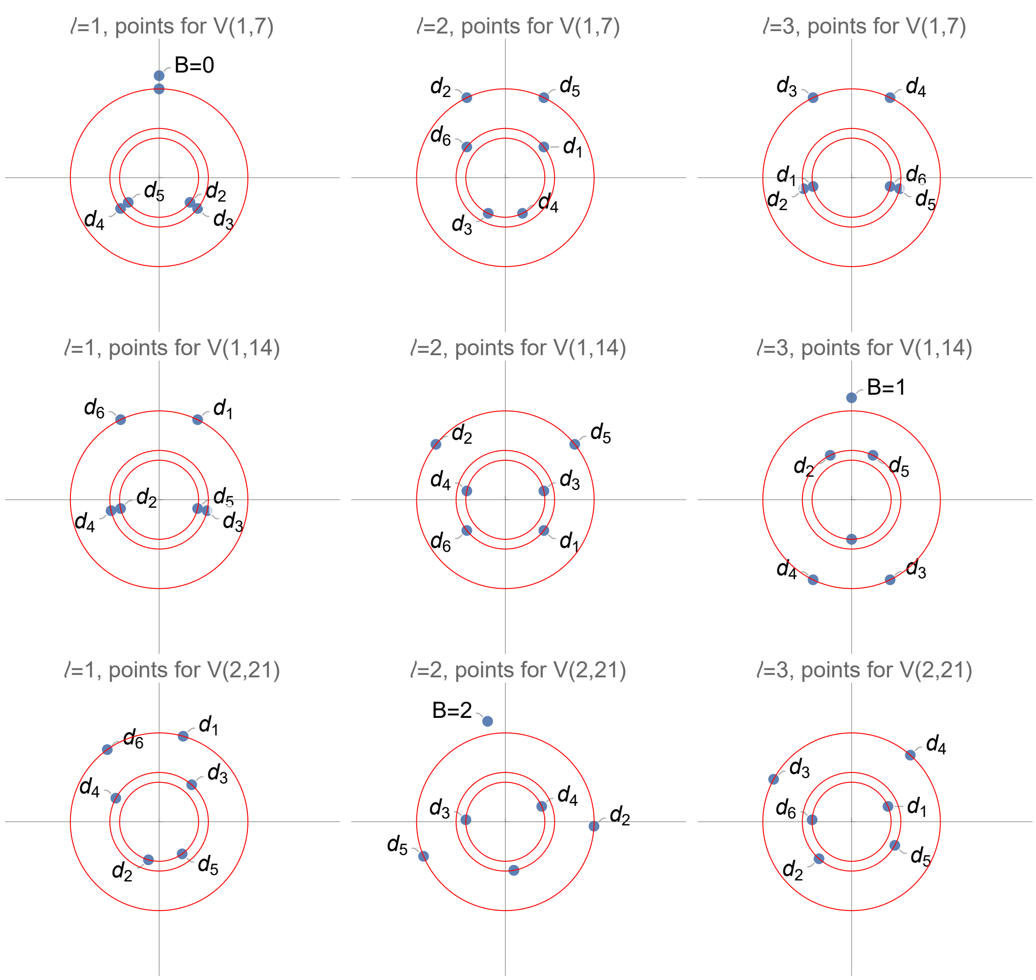
- $c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\operatorname{Arg}(Q \to P; 2) = \frac{1}{2}$;
- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\operatorname{Arg}(Q \to P; 3) = \frac{1}{2}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\operatorname{Arg}(Q \to P; 1) = 0$.

The following graphs for $c' \equiv 1, 2, 3 \pmod 7$ show the relative arguments of corresponding styles in Condition 6.5. In each graph, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.

$c' \equiv 4 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{2}$		$-\frac{3}{7}$		$-\frac{2}{7}$		$-\frac{3}{7}$		$\frac{1}{2}$		$\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{3}{14}$		$\frac{1}{2}$		$\frac{3}{7}$		$-\frac{1}{2}$		$-\frac{3}{14}$		0	
$\mathrm{Arg}(d_u \to d_v; 3)$		$\frac{3}{7}$		$\frac{3}{14}$		$\frac{2}{7}$		$\frac{3}{14}$		$\frac{3}{7}$		$\frac{3}{7}$	

$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{5}{14}$		$-\frac{3}{7}$		$\frac{3}{7}$		$-\frac{3}{7}$		$-\frac{5}{14}$		$\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{2}$		$\frac{5}{14}$		$-\frac{3}{7}$		$\frac{5}{14}$		$\frac{1}{2}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$\frac{3}{7}$		$\frac{1}{2}$		$\frac{1}{7}$		$\frac{1}{2}$		$\frac{3}{7}$		0	

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{5}{14}$		0		$\frac{2}{7}$		0		$\frac{5}{14}$		0	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{3}{14}$		$-\frac{5}{14}$		$-\frac{1}{7}$		$-\frac{5}{14}$		$-\frac{3}{14}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		0		$\frac{3}{14}$		$\frac{1}{7}$		$\frac{3}{14}$		0		$\frac{3}{7}$	



Visualizing by the above graphs, (6.11) is proved by the following trigonometric identities:

$$\begin{aligned}
&C_7^{2,4}\sin(\tfrac{\pi}{7})\left(\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{4,6}\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{6,2}\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{2,4}\sin(\tfrac{\pi}{7})\sqrt{7},\\
&C_7^{2,4}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{4,6}\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{6,2}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{1}{\sin(\frac{3\pi}{7})}\right)=-C_7^{6,2}\sin(\tfrac{3\pi}{7})\sqrt{7},\\
&C_7^{2,4}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{4,6}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{6,2}\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{4,6}\sin(\tfrac{2\pi}{7})\sqrt{7},
\end{aligned}$$

as well as the identities where the tuple $(C_7^{2,4}, C_7^{4,6}, C_7^{6,2})$ is changed to $(C_7^{4,6}, C_7^{6,2}, C_7^{2,4})$. This proves (7-1) of Theorem 1.3.

6.5. **(7-2) of Theorem 1.3.** We follow the definition (6.8), (6.6) and (6.7) but we use

$$P_3(d) := e\left(\frac{2d}{c}\right) \quad \text{and} \quad Q_3(B) := e\left(\frac{2B}{A}\right) \tag{6.12}$$

instead. The following equation suffices to prove (7-2):

$$C_7^{0,6} s_{r,c}^{(1)} + C_7^{0,2} s_{r,c}^{(2)} + C_7^{0,4} s_{r,c}^{(3)} = 0. \tag{6.13}$$

When $49|c$, there is no $Q(B)$ term. By subtracting $\frac{3}{7}$ from (6.4), we get $\operatorname{Arg}(d \to d_*; \ell) \neq 0$ when $r \equiv d \equiv 1, 3, 4, 6 \pmod 7$, hence $s_{r,c}^{(\ell)} = 0$ for $\ell = 1, 2, 3$. When $r \equiv d \equiv 2, 5 \pmod 7$ ($a \equiv 4, 3 \pmod 7$), note that $P_1(d) = (-1)^{(a+1)c\ell}/\sin(\frac{\pi a \ell}{7})$ has

$$\operatorname{sgn} P_1(d) = \begin{cases} 1, & \ell = 1, 2,\ a \equiv 3 \pmod 7;\ \text{or } \ell = 3,\ a \equiv 4 \pmod 7; \\ -1, & \ell = 1, 2,\ a \equiv 4 \pmod 7;\ \text{or } \ell = 3,\ a \equiv 3 \pmod 7. \end{cases}$$

By

$$C_7^{0,6} \frac{\sin(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})} + C_7^{0,2} \frac{\sin(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})} - C_7^{0,4} \frac{\sin(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})} = 0$$

We have proved (6.13) for $49|c$.

When $7\|c'$, from Condition 6.5 for the $7n+1$ case, we need to add $\pm\frac{\ell}{7}$ to $\operatorname{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ and add $\frac{\beta}{7}$ to $\operatorname{Arg}(d_j \to d_{j+1}; \ell)$ for $1 \le j \le 5$. We get the following condition.

Condition 6.6. *For the $7n+2$ case, we have the following conditions on* $\operatorname{Arg}(Q \to P; \ell)$ *when* $A\ell \equiv \pm 1 \pmod 7$ *with tables for* $\operatorname{Arg}_j(d_u \to d_v; \ell)$.

- $c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\operatorname{Arg}(Q \to P; 1) = \frac{1}{7}$;

$c' \equiv 1 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$-\frac{3}{14}$	$\frac{1}{7}$	$-\frac{1}{7}$	$\frac{1}{7}$	$-\frac{3}{14}$	$\frac{2}{7}$
$\operatorname{Arg}(d_u \to d_v; 2)$	$\frac{5}{14}$	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{5}{14}$	0
$\operatorname{Arg}(d_u \to d_v; 3)$	$\frac{1}{7}$	$-\frac{1}{14}$	0	$-\frac{1}{14}$	$\frac{1}{7}$	$-\frac{1}{7}$

- $c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\operatorname{Arg}(Q \to P; 3) = \frac{1}{14}$;

$c' \equiv 2 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$-\frac{1}{14}$	0	$\frac{1}{7}$	0	$-\frac{1}{14}$	0
$\operatorname{Arg}(d_u \to d_v; 2)$	$\frac{1}{14}$	$\frac{3}{14}$	0	$\frac{3}{14}$	$\frac{1}{14}$	$\frac{3}{7}$
$\operatorname{Arg}(d_u \to d_v; 3)$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{3}{7}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{1}{7}$

- $c' \equiv 3 \pmod 7$, $\beta = 5$. $A \cdot 2 = 7T - 1$, $\operatorname{Arg}(Q \to P; 2) = \frac{3}{14}$;

$c' \equiv 3 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\operatorname{Arg}(d_u \to d_v; 1)$	$\frac{3}{14}$	$\frac{1}{7}$	0	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{2}{7}$
$\operatorname{Arg}(d_u \to d_v; 2)$	$-\frac{1}{14}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{3}{14}$	$-\frac{1}{14}$	$\frac{3}{7}$
$\operatorname{Arg}(d_u \to d_v; 3)$	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{2}{7}$	0

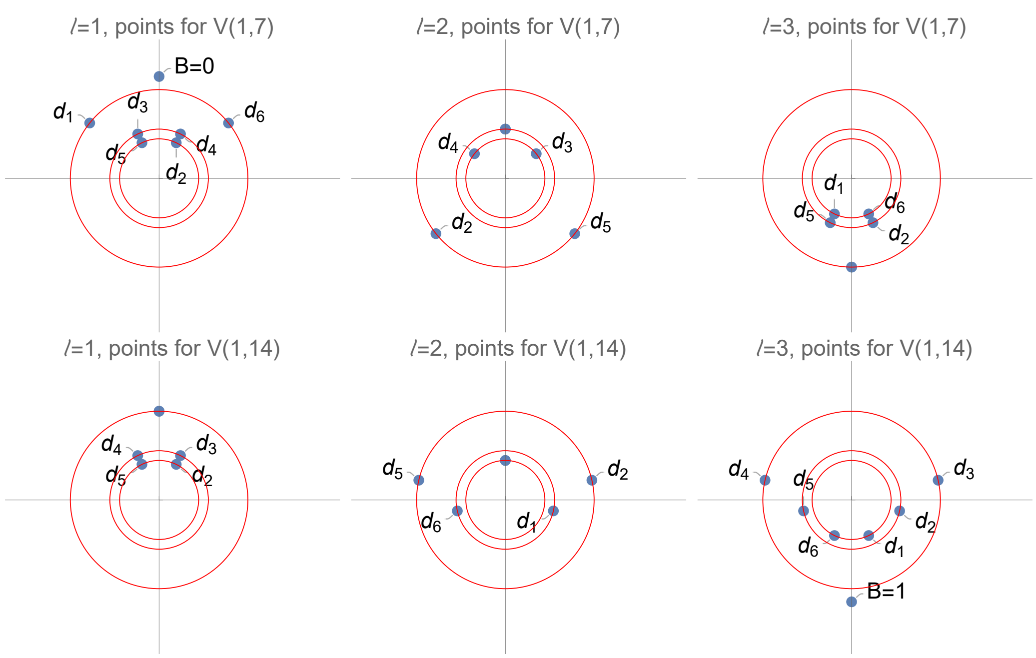
$c' \equiv 4 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{3}{14}$		$-\frac{1}{7}$		0		$-\frac{1}{7}$		$-\frac{3}{14}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{14}$		$-\frac{3}{14}$		$-\frac{2}{7}$		$-\frac{3}{14}$		$\frac{1}{14}$		$-\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{2}{7}$		$-\frac{1}{2}$		$-\frac{3}{7}$		$-\frac{1}{2}$		$-\frac{2}{7}$		0	

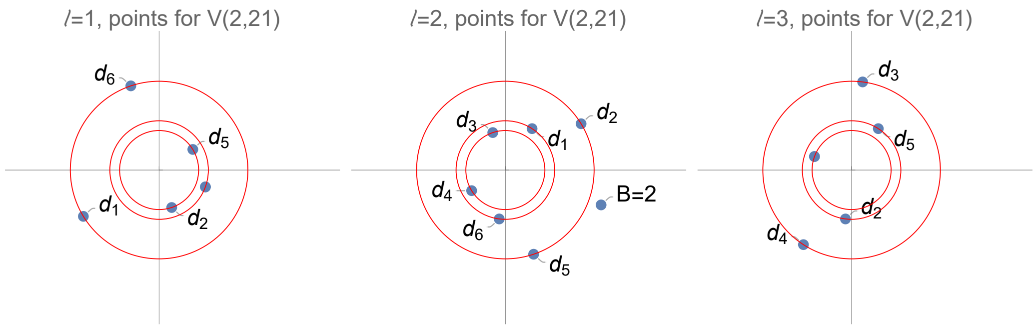
$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{14}$		0		$-\frac{1}{7}$		0		$\frac{1}{14}$		0	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{1}{14}$		$-\frac{3}{14}$		0		$-\frac{3}{14}$		$-\frac{1}{14}$		$-\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{1}{7}$		$-\frac{1}{14}$		$-\frac{3}{7}$		$-\frac{1}{14}$		$-\frac{1}{7}$		$-\frac{1}{7}$	

- $c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\mathrm{Arg}(Q \to P; 2) = -\frac{3}{14}$;
- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\mathrm{Arg}(Q \to P; 3) = -\frac{1}{14}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{1}{7}$.

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{3}{14}$		$-\frac{1}{7}$		$\frac{1}{7}$		$-\frac{1}{7}$		$\frac{3}{14}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{5}{14}$		$-\frac{1}{2}$		$-\frac{2}{7}$		$-\frac{1}{2}$		$-\frac{5}{14}$		0	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{1}{7}$		$\frac{1}{14}$		0		$\frac{1}{14}$		$-\frac{1}{7}$		$\frac{1}{7}$	

The following graphs for $c' \equiv 1, 2, 3 \pmod 7$ show the relative arguments of corresponding styles in Condition 6.5. In each graph, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.





Visualizing by the above graphs, (6.13) is proved by the following trigonometric identities:

$$C_7^{0,6}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,2}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)$$

$$+C_7^{0,4}\sin(\tfrac{3\pi}{7})\left(-\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,6}\sin(\tfrac{\pi}{7})\sqrt{7},$$

$$C_7^{0,6}\sin(\tfrac{\pi}{7})\left(-\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,2}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{1}{\sin(\frac{3\pi}{7})}\right)$$

$$+C_7^{0,4}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,4}\sin(\tfrac{3\pi}{7})\sqrt{7},$$

$$C_7^{0,6}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,2}\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)$$

$$+C_7^{0,4}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{1}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,2}\sin(\tfrac{2\pi}{7})\sqrt{7}.$$

This proves (7-2) of Theorem 1.3.

6.6. **(7-3) of Theorem 1.3.** We follow the definition (6.8), (6.6) and (6.7) but we use

$$P_3(d) := e\left(\frac{3d}{c}\right) \quad \text{and} \quad Q_3(B) := e\left(\frac{3B}{A}\right) \tag{6.14}$$

instead. The following equations suffice to prove (7-3):

$$\begin{aligned} C_7^{0,4} s_{r,c}^{(1)} + C_7^{0,6} s_{r,c}^{(2)} + C_7^{0,2} s_{r,c}^{(3)} &= 0, \\ C_7^{2,6} s_{r,c}^{(1)} + C_7^{4,2} s_{r,c}^{(2)} + C_7^{6,4} s_{r,c}^{(3)} &= 0. \end{aligned} \tag{6.15}$$

When $49|c$, there is no $Q(B)$ term. By subtracting $\frac{2}{7}$ from (6.4), $\mathrm{Arg}(d \to d_*; \ell)$ is always a non-zero constant for a fixed $r \pmod{c'}$. Then we get (6.15) by $s_{r,c}^{\ell} = 0$ for $\ell = 1, 2, 3$.

When $7\|c'$, from Condition 6.6 for the $7n+2$ case, we need to add $\pm\frac{\ell}{7}$ to $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ and add $\frac{\beta}{7}$ to $\mathrm{Arg}(d_j \to d_{j+1}; \ell)$ for $1 \le j \le 5$. We get the following condition.

Condition 6.7. *For the $7n+3$ case, we have the following conditions on* $\mathrm{Arg}(Q \to P; \ell)$ *when* $A\ell \equiv \pm 1 \pmod 7$ *with tables for* $\mathrm{Arg}_j(d_u \to d_v; \ell)$.

- $c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\mathrm{Arg}(Q \to P; 1) = \frac{2}{7}$;

$c' \equiv 1 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{1}{14}$		$\frac{2}{7}$		0		$\frac{2}{7}$		$-\frac{1}{14}$		$-\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{2}$		$-\frac{5}{14}$		$\frac{3}{7}$		$-\frac{5}{14}$		$\frac{1}{2}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$\frac{2}{7}$		$\frac{1}{14}$		$\frac{1}{7}$		$\frac{1}{14}$		$\frac{2}{7}$		$\frac{1}{7}$	

- $c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\mathrm{Arg}(Q \to P; 3) = -\frac{5}{14}$;

$c' \equiv 2 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{2}$		$-\frac{3}{7}$		$-\frac{2}{7}$		$-\frac{3}{7}$		$\frac{1}{2}$		$\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{5}{14}$		$-\frac{3}{14}$		$-\frac{3}{7}$		$-\frac{3}{14}$		$-\frac{5}{14}$		$-\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{2}{7}$		$-\frac{5}{14}$		0		$-\frac{5}{14}$		$-\frac{2}{7}$		$\frac{2}{7}$	

- $c' \equiv 3 \pmod 7$, $\beta = 5$. $A \cdot 2 = 7T - 1$, $\mathrm{Arg}(Q \to P; 2) = -\frac{1}{14}$;

$c' \equiv 3 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{1}{14}$		$-\frac{1}{7}$		$-\frac{2}{7}$		$-\frac{1}{7}$		$-\frac{1}{14}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{5}{14}$		$-\frac{1}{14}$		0		$-\frac{1}{14}$		$-\frac{5}{14}$		$-\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		0		$\frac{3}{14}$		$\frac{1}{7}$		$\frac{3}{14}$		0		$\frac{3}{7}$	

- $c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\mathrm{Arg}(Q \to P; 2) = \frac{1}{14}$;

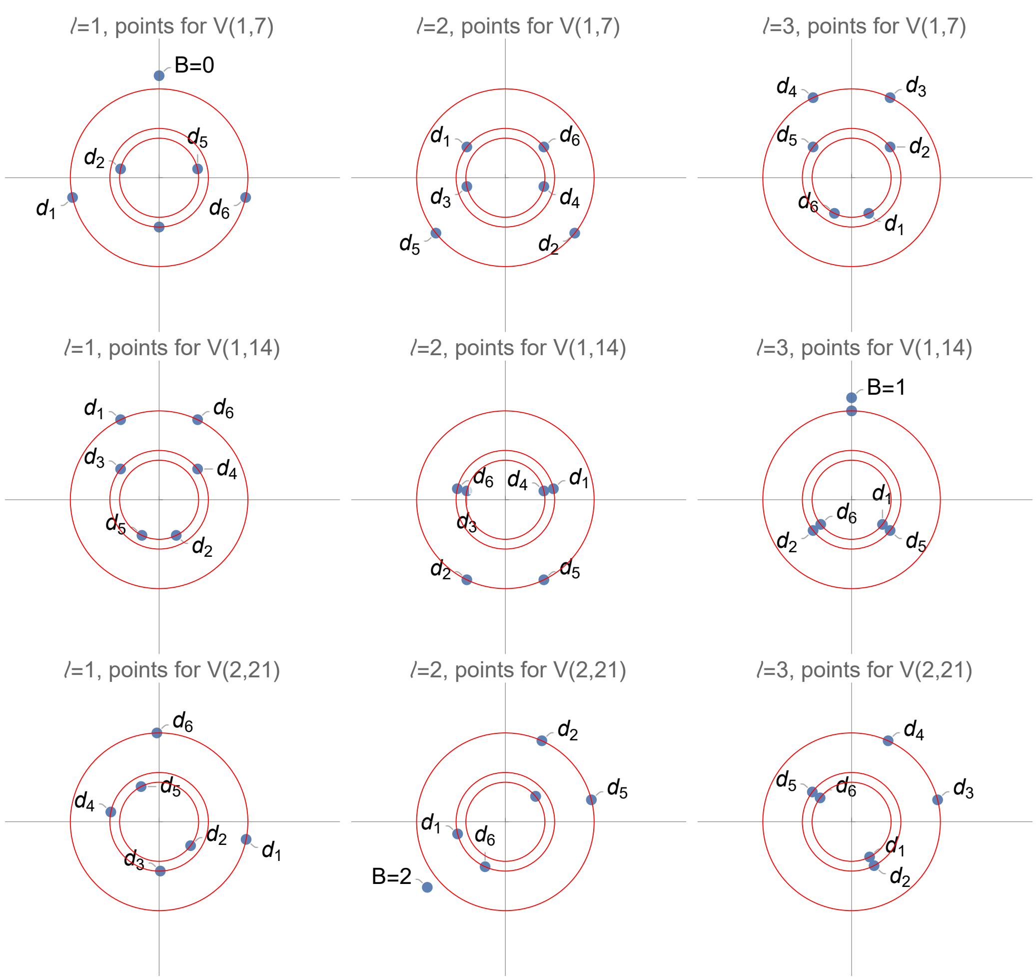
$c' \equiv 4 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{14}$		$\frac{1}{7}$		$\frac{2}{7}$		$\frac{1}{7}$		$\frac{1}{14}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{5}{14}$		$\frac{1}{14}$		0		$\frac{1}{14}$		$\frac{5}{14}$		$\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		0		$-\frac{3}{14}$		$-\frac{1}{7}$		$-\frac{3}{14}$		0		$-\frac{3}{7}$	

$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{2}$		$\frac{3}{7}$		$\frac{2}{7}$		$\frac{3}{7}$		$\frac{1}{2}$		$-\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{5}{14}$		$\frac{3}{14}$		$\frac{3}{7}$		$\frac{3}{14}$		$\frac{5}{14}$		$\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$\frac{2}{7}$		$\frac{5}{14}$		0		$\frac{5}{14}$		$\frac{2}{7}$		$-\frac{2}{7}$	

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$\frac{1}{14}$		$-\frac{2}{7}$		0		$-\frac{2}{7}$		$\frac{1}{14}$		$\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{1}{2}$		$\frac{5}{14}$		$-\frac{3}{7}$		$\frac{5}{14}$		$\frac{1}{2}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{2}{7}$		$-\frac{1}{14}$		$-\frac{1}{7}$		$-\frac{1}{14}$		$-\frac{2}{7}$		$-\frac{1}{7}$	

- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\mathrm{Arg}(Q \to P; 3) = \frac{5}{14}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{2}{7}$.

The following graphs for $c' \equiv 1, 2, 3 \pmod 7$ show the relative arguments of corresponding styles in Condition 6.5. In each graph, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.



Visualizing by the above graphs, (6.15) is proved by the following trigonometric identities:

$$
\begin{aligned}
&C_7^{0,4}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,6}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,2}\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,4}\sin(\tfrac{\pi}{7})\sqrt{7},\\
&C_7^{0,4}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,6}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,2}\sin(\tfrac{3\pi}{7})\left(\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,2}\sin(\tfrac{3\pi}{7})\sqrt{7},\\
&C_7^{0,4}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,6}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{1}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,2}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,6}\sin(\tfrac{2\pi}{7})\sqrt{7},
\end{aligned}
$$

as well as the identities where the tuple $(C_7^{0,4}, C_7^{0,6}, C_7^{0,2})$ is changed to $(C_7^{2,6}, C_7^{4,2}, C_7^{6,4})$. This proves (7-3) of Theorem 1.3.

6.7. **(7-4) of Theorem 1.3.** We follow the definition (6.8), (6.6) and (6.7) but we use

$$P_3(d) := e\left(\frac{4d}{c}\right) \quad \text{and} \quad Q_3(B) := e\left(\frac{4B}{A}\right) \tag{6.16}$$

instead. The following equations suffice to prove (7-4):

$$\begin{aligned} C_7^{0,2} s_{r,c}^{(1)} + C_7^{0,4} s_{r,c}^{(2)} + C_7^{0,6} s_{r,c}^{(3)} = 0, \\ C_7^{2,6} s_{r,c}^{(1)} + C_7^{4,2} s_{r,c}^{(2)} + C_7^{6,4} s_{r,c}^{(3)} = 0. \end{aligned} \tag{6.17}$$

When $49|c$, there is no $Q(B)$ term. By subtracting $\frac{1}{7}$ from (6.4), $\mathrm{Arg}(d \to d_*; \ell)$ is always a non-zero constant for a fixed $r \pmod{c'}$. Then we get (6.17) by $s_{r,c}^{(\ell)} = 0$ for $\ell = 1, 2, 3$.

When $7\|c'$, from Condition 6.7 for the $7n+3$ case, we need to add $\pm\frac{\ell}{7}$ to $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ and add $\frac{\beta}{7}$ to $\mathrm{Arg}(d_j \to d_{j+1}; \ell)$ for $1 \le j \le 5$. We get the following condition.

Condition 6.8. *For the $7n+4$ case, we have the following conditions on $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ with tables for $\mathrm{Arg}_j(d_u \to d_v; \ell)$.*

- *$c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\mathrm{Arg}(Q \to P; 1) = \frac{3}{7}$;*

$c' \equiv 1 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{1}{14}$	$\frac{3}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{1}{14}$	$-\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{5}{14}$	$-\frac{3}{14}$	$-\frac{3}{7}$	$-\frac{3}{14}$	$-\frac{5}{14}$	$-\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{3}{7}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{3}{14}$	$\frac{3}{7}$	$\frac{3}{7}$

- *$c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\mathrm{Arg}(Q \to P; 3) = \frac{3}{14}$;*

$c' \equiv 2 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{3}{14}$	$\frac{5}{14}$	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{3}{14}$	$-\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{2}{7}$	$\frac{3}{14}$	$-\frac{3}{7}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{3}{7}$

- *$c' \equiv 3 \pmod 7$, $\beta = 5$. $A \cdot 2 = 7T - 1$, $\mathrm{Arg}(Q \to P; 2) = -\frac{5}{14}$;*

$c' \equiv 3 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{5}{14}$	$-\frac{3}{7}$	$\frac{3}{7}$	$-\frac{3}{7}$	$-\frac{5}{14}$	$\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$\frac{5}{14}$	$\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{2}{7}$	$-\frac{1}{14}$	$-\frac{1}{7}$	$-\frac{1}{14}$	$-\frac{2}{7}$	$-\frac{1}{7}$

- *$c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\mathrm{Arg}(Q \to P; 2) = \frac{5}{14}$;*

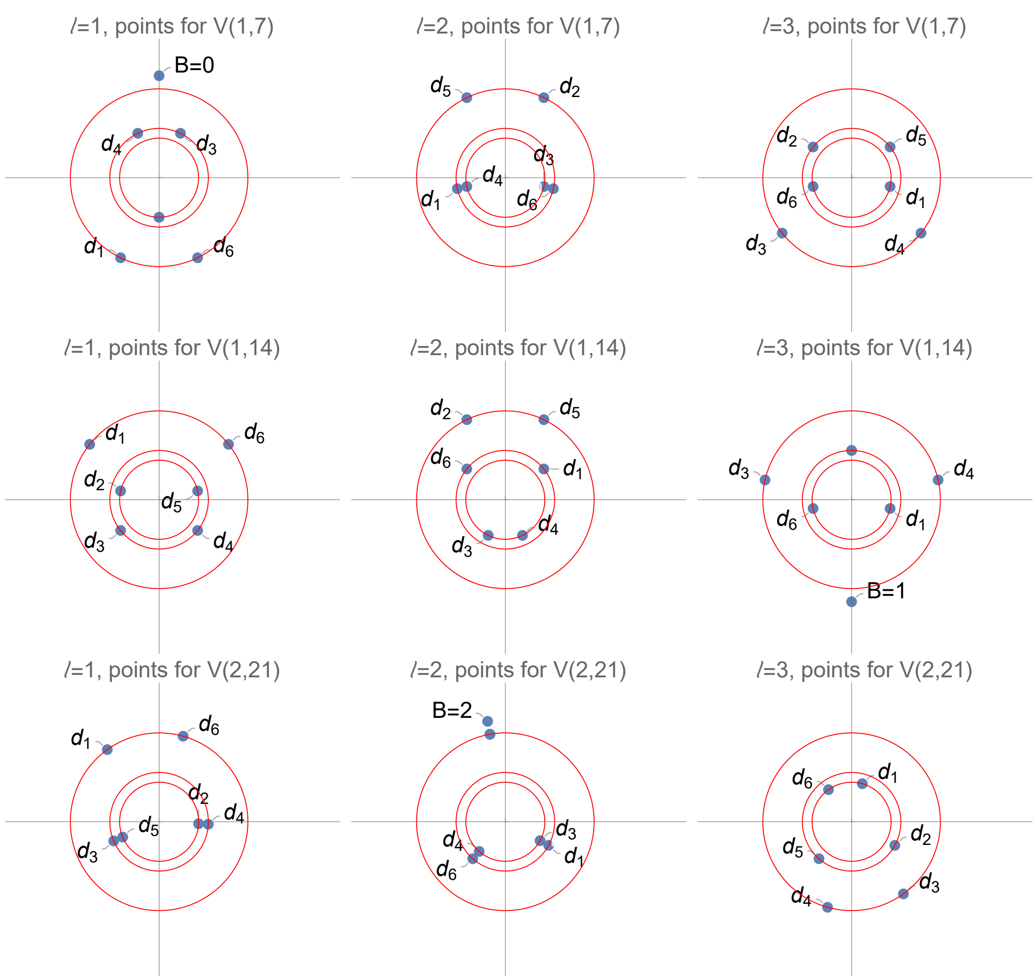
$c' \equiv 4 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{5}{14}$	$\frac{3}{7}$	$-\frac{3}{7}$	$\frac{3}{7}$	$\frac{5}{14}$	$-\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{5}{14}$	$\frac{5}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	$-\frac{5}{14}$	$-\frac{2}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{2}{7}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{1}{14}$	$\frac{2}{7}$	$\frac{1}{7}$

$c' \equiv 5 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{1}{14}$		$-\frac{1}{7}$		$-\frac{2}{7}$		$-\frac{1}{7}$		$-\frac{1}{14}$		$-\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$-\frac{3}{14}$		$-\frac{5}{14}$		$-\frac{1}{7}$		$-\frac{5}{14}$		$-\frac{3}{14}$		$\frac{2}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{2}{7}$		$-\frac{3}{14}$		$\frac{3}{7}$		$-\frac{3}{14}$		$-\frac{2}{7}$		$-\frac{3}{7}$	

$c' \equiv 6 \pmod 7$	d_1	$\to$	d_2	$\to$	d_3	$\to$	d_4	$\to$	d_5	$\to$	d_6	$\to$	d_1
$\mathrm{Arg}(d_u \to d_v; 1)$		$-\frac{1}{14}$		$-\frac{3}{7}$		$-\frac{1}{7}$		$-\frac{3}{7}$		$-\frac{1}{14}$		$\frac{1}{7}$	
$\mathrm{Arg}(d_u \to d_v; 2)$		$\frac{5}{14}$		$\frac{3}{14}$		$\frac{3}{7}$		$\frac{3}{14}$		$\frac{5}{14}$		$\frac{3}{7}$	
$\mathrm{Arg}(d_u \to d_v; 3)$		$-\frac{3}{7}$		$-\frac{3}{14}$		$-\frac{2}{7}$		$-\frac{3}{14}$		$-\frac{3}{7}$		$-\frac{3}{7}$	

- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\mathrm{Arg}(Q \to P; 3) = -\frac{3}{14}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{3}{7}$.

The following graphs for $c' \equiv 1, 2, 3 \pmod 7$ show the relative arguments of corresponding styles in Condition 6.5. In each graph, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.



Visualizing by the above graphs, (6.17) is proved by the following trigonometric identities:

$$
\begin{aligned}
&C_7^{0,2}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{1}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,4}\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,6}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,2}\sin(\tfrac{\pi}{7})\sqrt{7},\\
&C_7^{0,2}\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,4}\sin(\tfrac{2\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,6}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{1}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,6}\sin(\tfrac{3\pi}{7})\sqrt{7},\\
&C_7^{0,2}\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{3\pi}{7})}\right)+C_7^{0,4}\sin(\tfrac{2\pi}{7})\left(\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\qquad+C_7^{0,6}\sin(\tfrac{3\pi}{7})\left(-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=-C_7^{0,4}\sin(\tfrac{2\pi}{7})\sqrt{7},
\end{aligned}
$$

as well as the identities where the tuple $(C_7^{0,2},C_7^{0,4},C_7^{0,6})$ is changed to $(C_7^{2,6},C_7^{4,2},C_7^{6,4})$. This proves (7-4) of Theorem 1.3.

6.8. **(7-6) of Theorem 1.3.** We follow the definition (6.8), (6.6) and (6.7) but we use

$$P_3(d) := e\left(\frac{6d}{c}\right) \quad \text{and} \quad Q_3(B) := e\left(\frac{6B}{A}\right) \tag{6.18}$$

instead. The following equations suffice to prove (7-6):

$$\left(C_7^{0,4} + C_7^{2,6}\right) s_{r,c}^{(1)} + \left(C_7^{0,6} + C_7^{4,2}\right) s_{r,c}^{(2)} + \left(C_7^{0,2} + C_7^{6,4}\right) s_{r,c}^{(3)} = 0. \tag{6.19}$$

When $49|c$, there is no $Q(B)$ term. By adding $\frac{1}{7}$ from (6.4), $\mathrm{Arg}(d \to d_*; \ell)$ is a non-zero constant for $r \equiv d \equiv 1, 2, 5, 6 \pmod 7$, which shows $s_{r,c}^{(\ell)} = 0$ for $\ell = 1, 2, 3$ and proves (6.19). When $r \equiv d \equiv 3, 4 \pmod 7$, note that $P_1(d) = (-1)^{(a+1)c\ell}/\sin(\frac{\pi a \ell}{7})$ has

$$\operatorname{sgn} P_1(d) = \begin{cases} 1, & \ell = 1, 3,\ a \equiv 5 \pmod 7;\ \text{or } \ell = 2,\ a \equiv 2 \pmod 7; \\ -1, & \ell = 1, 3,\ a \equiv 2 \pmod 7;\ \text{or } \ell = 2,\ a \equiv 5 \pmod 7. \end{cases}$$

By

$$(C_7^{0,4} + C_7^{2,6}) \frac{\sin(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})} - (C_7^{0,6} + C_7^{4,2}) \frac{\sin(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})} + (C_7^{0,2} + C_7^{6,4}) \frac{\sin(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})} = 0,$$

we finish the proof of (6.19) when $49|c$.

When $7\|c'$, from Condition 6.8 for the $7n+4$ case, we need to add $\pm\frac{2\ell}{7}$ to $\mathrm{Arg}(Q \to P; \ell)$ when $A\ell \equiv \pm 1 \pmod 7$ and add $\frac{2\beta}{7}$ to $\mathrm{Arg}(d_j \to d_{j+1}; \ell)$ for $1 \le j \le 5$. (One may also begin from Condition 6.3 for the $7n+5$ case.)

We get the following condition.

Condition 6.9. *For the $7n+6$ case, we have the following conditions on* $\mathrm{Arg}(Q \to P; \ell)$ *when* $A\ell \equiv \pm 1 \pmod 7$ *with tables for* $\mathrm{Arg}_j(d_u \to d_v; \ell)$.

- $c' \equiv 1 \pmod 7$, $\beta = 1$. $A \cdot 1 = 7T + 1$, $\mathrm{Arg}(Q \to P; 1) = -\frac{2}{7}$;

$c' \equiv 1 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{5}{14}$	$-\frac{2}{7}$	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{5}{14}$	$\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{1}{7}$	$\frac{1}{14}$	$-\frac{1}{14}$	$\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{2}{7}$	$\frac{1}{2}$	$-\frac{3}{7}$	$\frac{1}{2}$	$-\frac{2}{7}$	0

- $c' \equiv 2 \pmod 7$, $\beta = 4$. $A \cdot 3 = 7T - 1$, $\mathrm{Arg}(Q \to P; 3) = \frac{5}{14}$;

$c' \equiv 2 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{3}{14}$	$-\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{5}{14}$	$\frac{1}{2}$	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{5}{14}$	0
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{3}{7}$	$\frac{5}{14}$	$-\frac{2}{7}$	$\frac{5}{14}$	$\frac{3}{7}$	$-\frac{2}{7}$

- $c' \equiv 3 \pmod 7$, $\beta = 5$. $A \cdot 2 = 7T - 1$, $\mathrm{Arg}(Q \to P; 2) = \frac{1}{14}$;
- $c' \equiv 4 \pmod 7$, $\beta = 2$. $A \cdot 2 = 7T + 1$, $\mathrm{Arg}(Q \to P; 2) = -\frac{1}{14}$;
- $c' \equiv 5 \pmod 7$, $\beta = 3$. $A \cdot 3 = 7T + 1$, $\mathrm{Arg}(Q \to P; 3) = \frac{5}{14}$;
- $c' \equiv 6 \pmod 7$, $\beta = 6$. $A \cdot 1 = 7T - 1$, $\mathrm{Arg}(Q \to P; 1) = \frac{2}{7}$.

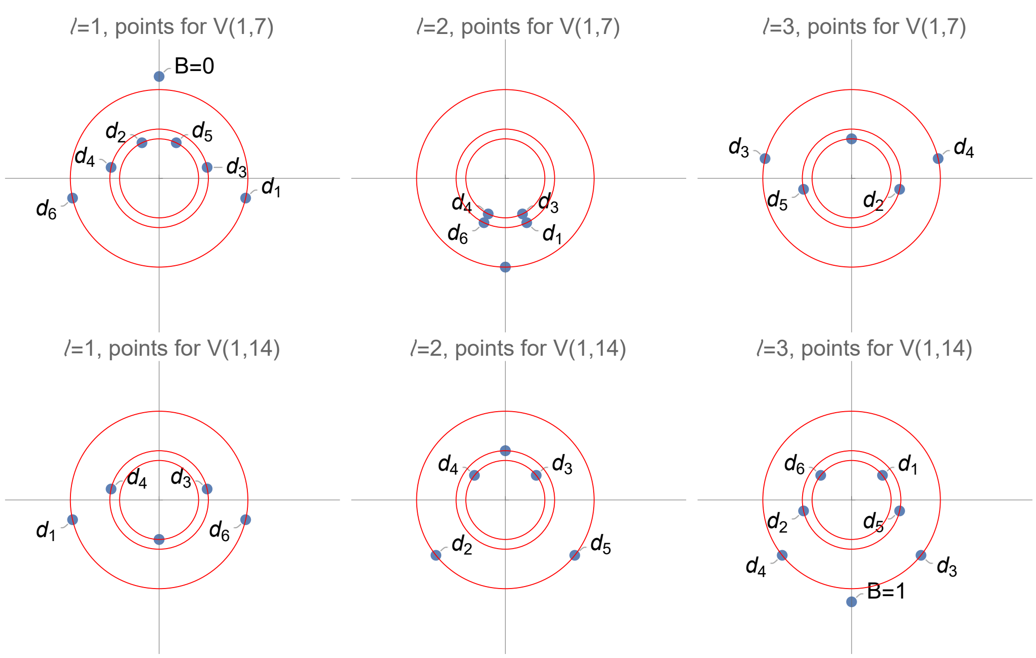
The following graphs for $c' \equiv 1, 2, 3 \pmod 7$ show the relative arguments of corresponding styles in Condition 6.5. In each graph, if d_u and d_v are not shown, then $P(d_u) = P(d_v)$ are both the remaining non-labeled point.

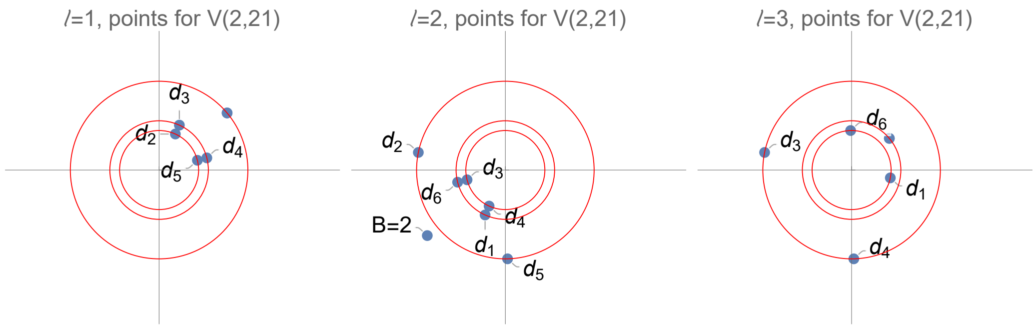
$c' \equiv 3 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$\frac{1}{14}$	0	$-\frac{1}{7}$	0	$\frac{1}{14}$	0
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{3}{14}$	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{1}{14}$	$-\frac{3}{14}$	$\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{1}{7}$	$\frac{5}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	$\frac{1}{7}$	$-\frac{2}{7}$

$c' \equiv 4 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{1}{14}$	0	$\frac{1}{7}$	0	$-\frac{1}{14}$	0
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{3}{14}$	$-\frac{1}{14}$	$-\frac{1}{7}$	$-\frac{1}{14}$	$\frac{3}{14}$	$-\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{1}{7}$	$-\frac{5}{14}$	$-\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{1}{7}$	$\frac{2}{7}$

$c' \equiv 5 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{3}{14}$	$-\frac{2}{7}$	$-\frac{3}{7}$	$-\frac{2}{7}$	$-\frac{3}{14}$	$\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$-\frac{5}{14}$	$\frac{1}{2}$	$-\frac{2}{7}$	$\frac{1}{2}$	$-\frac{5}{14}$	0
$\mathrm{Arg}(d_u \to d_v; 3)$	$-\frac{3}{7}$	$-\frac{5}{14}$	$\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{3}{7}$	$\frac{2}{7}$

$c' \equiv 6 \pmod 7$	$d_1 \to d_2$	$d_2 \to d_3$	$d_3 \to d_4$	$d_4 \to d_5$	$d_5 \to d_6$	$d_6 \to d_1$
$\mathrm{Arg}(d_u \to d_v; 1)$	$-\frac{5}{14}$	$\frac{2}{7}$	$-\frac{3}{7}$	$\frac{2}{7}$	$-\frac{5}{14}$	$-\frac{3}{7}$
$\mathrm{Arg}(d_u \to d_v; 2)$	$\frac{1}{14}$	$-\frac{1}{14}$	$\frac{1}{7}$	$-\frac{1}{14}$	$\frac{1}{14}$	$-\frac{1}{7}$
$\mathrm{Arg}(d_u \to d_v; 3)$	$\frac{2}{7}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{2}{7}$	0





Visualizing by the above graphs, (6.19) is proved by the following trigonometric identities:

$$\begin{aligned}
&(C_7^{0,4}+C_7^{2,6})\sin(\tfrac{\pi}{7})\left(-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}+\sqrt{7}\right)\\
&\quad+(C_7^{0,6}+C_7^{4,2})\sin(\tfrac{2\pi}{7})\left(-\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\quad+(C_7^{0,2}+C_7^{6,4})\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{1}{\sin(\frac{3\pi}{7})}\right)=0,\\
&(C_7^{0,4}+C_7^{2,6})\sin(\tfrac{\pi}{7})\left(\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{1}{\sin(\frac{3\pi}{7})}\right)\\
&\quad+(C_7^{0,6}+C_7^{4,2})\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{1}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\quad+(C_7^{0,2}+C_7^{6,4})\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{3\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}+\sqrt{7}\right)=0,\\
&(C_7^{0,4}+C_7^{2,6})\sin(\tfrac{\pi}{7})\left(-\frac{1}{\sin(\frac{\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}\right)\\
&\quad+(C_7^{0,6}+C_7^{4,2})\sin(\tfrac{2\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{2\pi}{7})}+\frac{\cos(\frac{\pi}{7})}{\sin(\frac{3\pi}{7})}+\sqrt{7}\right)\\
&\quad+(C_7^{0,2}+C_7^{6,4})\sin(\tfrac{3\pi}{7})\left(\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{\pi}{7})}-\frac{1}{\sin(\frac{2\pi}{7})}-\frac{\cos(\frac{2\pi}{7})}{\sin(\frac{3\pi}{7})}\right)=0.
\end{aligned}$$

This proves (7-6) of Theorem 1.3.

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